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EQUATIONS WITH THE HELP OF A DIGITAL COMPUTER

Paul C. Cromwell

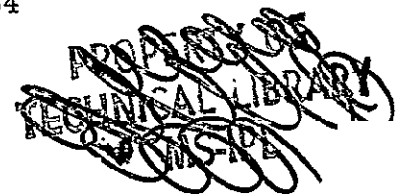
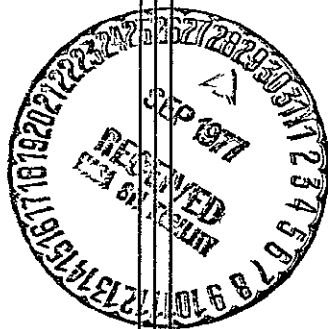
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George C. Marshall Space Flight Center
National Aeronautics and Space Administration

Huntsville, Alabama

January 31, 1964



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EQUATIONS WITH THE HELP OF A DIGITAL COMPUTER

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ABSTRACT

Analytic solutions for non-linear differential equations are difficult, time consuming and largely impractical for reasonably large values of the independent variable. The purpose of this study was to develop a technique for analytic (algebraic) solutions of autonomous and nonautonomous equations of the type

$$\sum_{n=0}^q c_n \frac{d^n}{dt^n} f(t) = g(t) + kN [f, f', f'' \text{ etc.}]$$

with the help of a digital computer.

In the equation n is an integer and q is a small integer. c_n is a constant. $g(t)$ is a forcing function. $N [f, f', f'' \text{ etc.}]$ is the non-linear term while k is the usual "small" parameter. N does not contain the independent variable t (time) explicitly. $f(t)$ is a continuous bounded function with finite initial conditions.

Two operational transform techniques have been programmed for the solution of equations of this type. To develop computer techniques only relatively simple non-linear differential equations have been considered. The theorem of Poincaré assures the convergence in all cases considered for "small" values of k .

In the few cases considered it has been possible to assimilate the secular terms into the solutions.

For cases where $f(t)$ is not a bounded function a direct series solution is developed which can be shown to be an analytic function.

All solutions have been checked against results obtained by numerical integration for given initial conditions and constants.

While the results of this study may be regarded as experimental in character it seems evident that at least certain types of non-linear differential equations not only can be solved with the help of a digital computer but that except for quite elementary equations must be solved in this way.

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1. INTRODUCTION

Engineers and physicists have been interested in non-linear problems for many years. While graphical methods or hand-calculated values for numerical integration could sometimes be employed, the usefulness of such solutions was largely restricted by the tedious calculations and techniques. With the advent of digital and analog computers numerical and graphical solutions could be obtained for most engineering problems. Such solutions require definite numerical values for design constants and initial conditions and many different runs are necessary to establish the character of the solution.

In many cases analytic (algebraic) solutions might prove quite helpful if not too complicated since the effect of changing design constants or initial conditions could be recognized from the form of the solution. This would be particularly true if the results could be approximated with two or three terms. Unfortunately analytic solutions for non-linear differential equations are both difficult and time consuming and, in most cases, quite impractical for large values of the independent variable.

During the past seven years the author and his students conducted a series of investigations leading to analytic solutions for non-linear and linear differential equations using a digital computer. Madhu¹ showed that it was possible to solve a non-linear differential equation using a power series in the independent variable, time. This approach was both cumbersome and unsuited to machine manipulation. It was evident that if machine calculations were to be used an operational or transform type of solution would be required.

Creecraft² developed solutions for several non-linear equations but only for cases where the solutions were known. Shiva³ showed that a linear differential equation with a variable coefficient could also be solved by a technique which could be adapted to computer operations. Regan⁴ programmed a computer for certain derivative operations and Erkanli⁵, Cheng⁷ and Wang⁸ developed tables of transforms for special

functions. Numerical solutions for the van der Pol equation were obtained by Chen⁶ while numerical solutions for the Mathieu equation were computed by Lee⁹.

The purpose of this report is to present the results of an experimental study in the use of a digital computer to obtain analytic solutions for certain types of non-linear differential equations. The solutions are limited to bounded analytic functions and convergence is assured by Poincaré's Theorem¹⁰. Appropriate cases include many of the equations usually discussed in text-books such as certain forms of the van der Pol and Duffing equations.

One section is devoted to direct series solutions where the results are unbounded analytic functions.

It is sometimes said that calculations which can be done by hand can always be programmed on a digital computer. Unfortunately it is not unusual to find that the whole procedure must be revised to carry through a given problem. In the case of non-linear solutions the powerful perturbation method would be very difficult to program due to the series of solutions required for the unknown coefficients. An operational type of solution seemed advisable since a computer could easily be programmed to perform the required operations. It should be emphasized that the Laplace Transform is employed only in evaluating the equation as presented. In particular the solution of several simultaneous non-linear differential equations by Laplace Transforms is not involved.

Two different techniques are described. The first type is closely associated with the development of a Maclaurin's series by repeated differentiations while the second type evaluates the time function directly.

A discussion of the conditions for convergence of the operational solution will be followed by an example presented in some detail.

An outline of the computer operations is then followed by a number of examples which show the power and weakness of the method. In each case the numerical values are substituted in the analytic solution to obtain a graphical solution which is then compared with a solution obtained by numerical integration.

2. TYPE OF EQUATION CONSIDERED

The primary purpose of this study was to develop a method for obtaining analytic solutions for non-linear differential equations using a digital computer. For this reason the equations selected were as simple as possible.

All equations are of the form:

$$\sum_{n=0}^q c_n \frac{d^n}{dt^n} f(t) = g(t) + kN[f, f', f'', \text{etc.}] \quad (1)$$

Here n is an integer and q is a small integer. c_n is a constant. $g(t)$ is a forcing function. $N[f, f', f'', \text{etc.}]$ is the non-linear term in polynomial form while k is the usual "small" parameter. The equation is autonomous if the forcing function is a constant and nonautonomous if it is not a constant. $f(t)$ is a bounded continuous single valued analytic function. All terms are limited to functions which may be developed in Taylor's series.

Equation (1) is the form usually considered in text books on non-linear differential equations and convergence of the solution is assured if the parameter k is small enough on the basis of the theorem of Poincaré¹⁰.

All solutions are therefore analytic functions.

3. CONVERGENCE OF THE OPERATIONAL SOLUTION

Consider a function of the type

$$f(t) = \sum_{m=0}^{\infty} k_m f_m(t) \quad (2)$$

$f(t)$ is a bounded, continuous, single valued function uniformly convergent in some closed interval $0 \leq t < h$, $h > 0$. All $f_m(t)$ are similarly defined (See Reference 11). In addition

$$f(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{n!},$$

a restriction required by the technique in the solution.

The Laplace direct transform is

$$F(p) = p \int_0^{\infty} f(t) e^{-pt} dt \quad (3)$$

where p is a complex number with real part greater than zero.

Equation (3) is to be carried through by integration by parts.

Since all $f_m(t)$ are bounded functions the upper limit contributes nothing to the final result:

$$F_m(p) = p \int_0^{\infty} f_m(t) e^{-pt} dt = \sum_{n=0}^{\infty} \frac{f_m^{(n)}(0)}{p^n}. \quad (4)$$

The result in p^{-n} is always absolutely and uniformly convergent in p (See Reference 12).

If the upper limit on the integral in (4) is removed the result is an "open-end integral" and the result in (4) may then be obtained by a series of differentiations carried through by a computer.

When all $f_m(t)$ converge uniformly and the integral in (4) converges

$$\begin{aligned} p \int_0^{\infty} f_m(t) e^{-pt} dt &= p \sum_{n=0}^{\infty} \int_0^{\infty} f_m^{(n)}(0) \frac{t^n}{n!} e^{-pt} dt \\ &= \sum_{n=0}^{\infty} \frac{f_m^{(n)}(0)}{p^n} = F_m(p). \end{aligned} \quad (5)$$

That is, the function and the series in t^n have identical transforms and all $F_m(p)$ are absolutely and uniformly convergent; $f_m(t)$ is in terms of a Maclaurin's series.

Now uniformly converging series can be summed (Reference 11):

$$F(p) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{k_m f_m^{(n)}(0)}{p^n}. \quad (6)$$

If (6) is now inserted in the inverse transform

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{F(p)}{p} e^{pt} dp \quad (7)$$

each of the p^{-n} th terms may be evaluated separately and the results added to obtain

$$f(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{k_m f_m^{(n)}(0) t^n}{n!}, \quad (8)$$

Data comes from the machine in the form

$$F_m(p) \sim \sum_{m=0}^q \sum_{n=0}^r \frac{k_m f_m^{(n)}(0)}{p^n} \quad (9)$$

where q and r are integers. Normally $q < 6$ and $r < 30$. The symbol $\sim$ indicates an approximation.

The direct series result equivalent to (8) is then

$$f(t) \sim \sum_{m=0}^q \sum_{n=0}^r \frac{k_m f_m^{(n)}(0) t^n}{n!} . \quad (10)$$

The series (8) is a convergent series by definition and since all $f_m(t)$ are uniformly convergent (10) is always the first portion of a converging series which may be carried as far as necessary.

If the raw data of (9) are sorted on k_m and the p^{-n} th terms summed in terms of known functions,

$$F(p) \sim \sum_{m=0}^q k_m F_m(p). \quad (11)$$

If these $F_m(p)$ are now evaluated by the inverse transform,

$$f(t) \sim \sum_{m=0}^q k_m f_m(t). \quad (12)$$

Since $f(t)$ and all $f_m(t)$ are bounded functions and the series is uniformly convergent, $f(t)$ converges if $q = \infty$.

Equation (12) is a "summation solution".

Such summation solutions are much more effective than the direct series solutions in producing results.

In the direct series solution all data are evaluated directly. It is not necessary to recognize the $F_m(p)$ in order to obtain a direct series solution.

On the other hand, a summation solution uses a few terms to recognize an operational term and the resulting solution converges to a usable result for much larger values of t than the direct series type of solution.

As might be expected summation solutions quite frequently have secular terms. These terms can be assimilated into the solution in some cases. With a computer presumably a sufficient number of them could be

obtained so that the series thus formed would be a good approximation for the unknown function over the desired range. In this case the condition of uniform convergence is not usually valid.

All summation solutions do not have secular terms, and in at least one instance, the solution for the pendulum, the frequency of oscillation can be predicted and no secular terms appear. This suggests that with proper summation of terms $A_n \left\{ f(0), f'(0) \text{ etc.}, k \right\} \cdot f_n \left\{ f(0), f'(0), \text{ etc.}, k, t \right\}$ where the coefficients A_n and the time functions contain k and the boundary conditions the results are the same as for Lindstedt's method¹³.

Operational solutions for equations defined in Section 2 are formal solutions of the equations. Convergence is assured for bounded functions if the parameter k is small enough and a sufficient number of terms is available.

In the special case where the formal solution $f(t)$ can be recognized as a bounded, continuous, single valued function uniformly convergent in some closed interval $0 < t < h$, $h > 0$ and k is small enough the result converges. Such results are analytic functions.

4. DERIVATIVE OPERATIONAL METHOD

Two operational methods have been developed for the solution of suitable non-linear differential equations using a computer. The "derivative" or "open-end integral" method is explained in this section by the solution of a well known first order equation. Section 5 explains the operations for the "expansion" method.

Consider the equation

$$f' + Af = -Bf^3 \quad (13)$$

where A and B are positive constants. B is "small". Physically one may visualize this equation as non-linear braking for a rotating mass. The solution $f(t)$ is therefore a bounded function.

Rewrite (13) as

$$f' = -Af - Bf^3 \quad (14)$$

and let $f(0) = a$, the only initial condition.

The direct transform is

$$F(p) = p \int_0^{\infty} f(t) e^{-pt} dt. \quad (15)$$

Integrating by parts:

$$\begin{aligned} F(p) &= p \left[\frac{-fe^{-pt}}{p} \Big|_0^{\infty} + \frac{1}{p} \int_0^{\infty} f' e^{-pt} dt \right] \\ &= a + \int_0^{\infty} f' e^{-pt} dt. \end{aligned} \quad (16)$$

Since the function $f(t)$ is bounded the upper limit in (15) never contributes to the final result and the integration by parts turns into a series of differentiations with the lower limit $f(0) = a$ inserted.

Continuing the integration by parts after substituting (14) in (16)

$$\begin{aligned}
F(p) &= a + \int_0^{\infty} (-Af - Bf^3) e^{-pt} dt \\
&= a - A \int_0^{\infty} f e^{-pt} dt \\
&\quad - B \int_0^{\infty} f^3 e^{-pt} dt \\
&= a - \frac{Aa}{p} + \frac{A}{p} \int_0^{\infty} f' e^{-pt} dt - \frac{Ba^3}{p} \\
&\quad + \frac{B}{p} \int_0^{\infty} 3f^2 f' e^{-pt} dt.
\end{aligned} \tag{17}$$

This procedure can easily be programmed on a computer using a defined operation which may be designated as the "open-end integral":

$$\begin{aligned}
F(p) &= p \int_0^{\infty} f(t) e^{-pt} dt \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{p^n}
\end{aligned} \tag{18}$$

where $f^{(n)}(t)$ is the n th derivative.

If the inverse transform is used term by term

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) t^n}{n!} \tag{19}$$

which is Maclaurin's series for $f(t)$.

Continuing with the derivative operations with Equation (18) the computer gives, for the first seven derivative operations:

$$\begin{aligned}
F(p) = a - & \frac{aA + a^3B}{p} + \frac{aA^2 + 4a^3AB + 3a^5B^2}{p^2} \\
& - \frac{aA^3 + 13a^3A^2B + 27a^5AB^2 + 15a^7B^3}{p^3} \\
& + \frac{aA^4 + 40a^3A^3B + 174a^5A^2B^2 + 240a^7AB^3 + 105a^9B^4}{p^4} \\
& - \frac{\left[aA^5 + 121a^3A^4B + 990a^5A^3B^2 + 2550a^7A^2B^3 + 2625a^9AB^4 \right. \\
& \quad \left. + 945a^{11}B^5 \right]}{p^5} \\
& + \frac{\left[aA^6 + 364a^3A^5B + 5313a^5A^4B^2 + 22,880a^7A^3B^3 \right. \\
& \quad \left. + 41,475a^9A^2B^4 + 34,020a^{11}AB^5 + 10,395a^{13}B^6 \right]}{p^6} \\
& - \frac{\left[aA^7 + 1093a^3A^6B + 27,657a^5A^5B^2 + 186,165a^7A^4B^3 \right. \\
& \quad + 532,875a^9A^3B^4 + 747,495a^{11}A^2B^5 + 509,355a^{13}AB^6 \\
& \quad \left. + 135,135a^{15}B^7 \right]}{p^7} \\
& + + + +
\end{aligned} \tag{20}$$

The Maclaurin's series equivalent to (19) is:

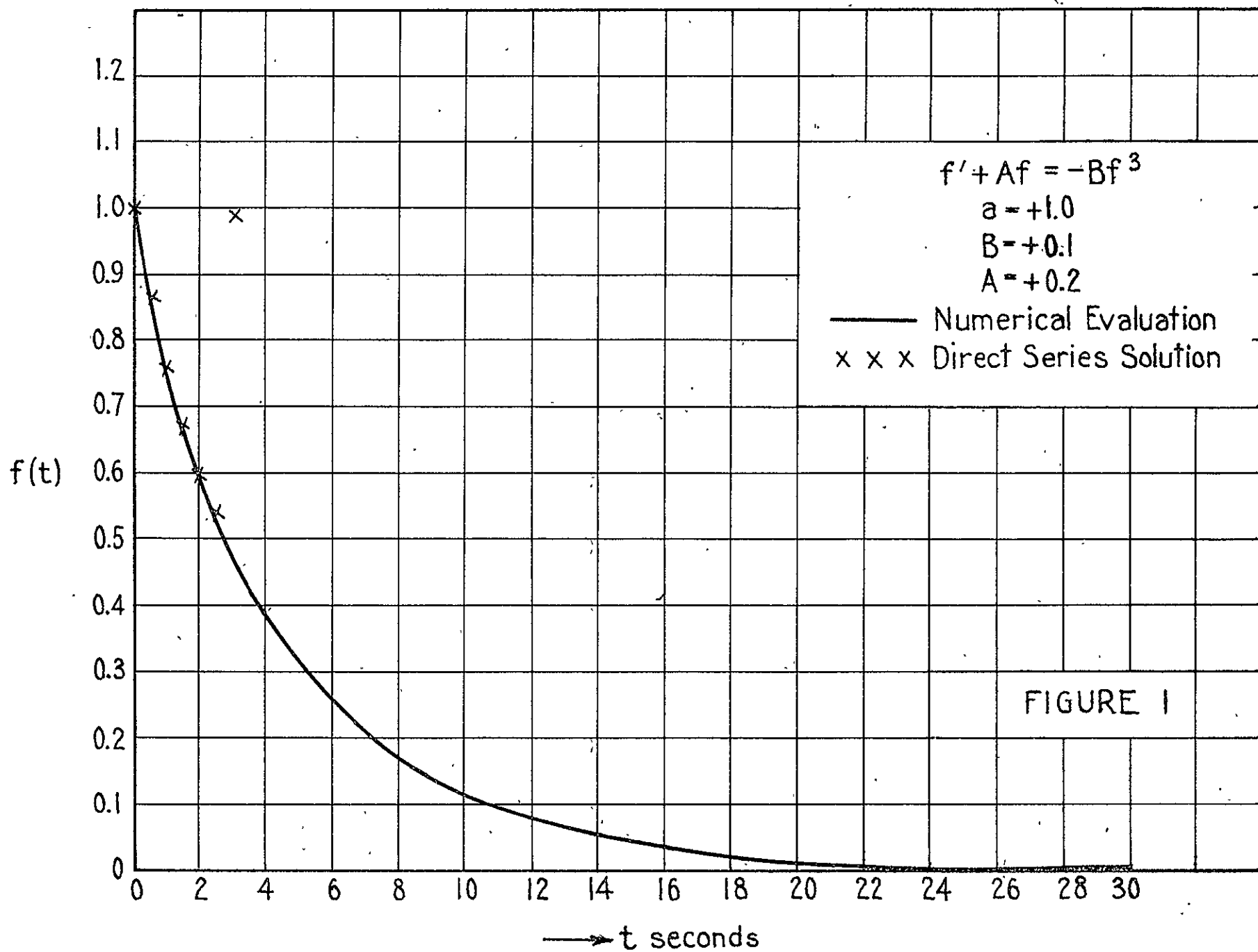
$$\begin{aligned}
f(t) = a - & \frac{(aA + a^3B)t}{1!} + \frac{(aA^2 + 4a^3AB + 3a^5B^2)t^2}{2!} \\
& - \frac{(aA^3 + 13a^3A^2B + 27a^5AB^2 + 15a^7B^3)t^3}{3!}
\end{aligned}$$

$$\begin{aligned}
& + \frac{(aA^4 + 40a^3A^3B + 174a^5A^2B^2 + 240a^7AB^3 + 105a^9B^4)t^4}{4!} \\
& - \frac{(aA^5 + 121a^3A^4B + 990a^5A^3B^2 + 2550a^7A^2B^3 + 2625a^9AB^4 \\
& + 945a^{11}B^5)t^5}{5!} \\
& + \frac{(aA^6 + 364a^3A^5B + 531a^5A^4B^2 + 22,800a^7A^3B^3 \\
& + 41,475a^9A^2B^4 + 34,020a^{11}AB^5 + 10,395a^{13}B^6)t^6}{6!} \\
& - \frac{(aA^7 + 1093a^3A^6B + 27,657a^5A^5B^2 + 186,165a^7A^4B^3 \\
& + 532,875a^9A^3B^4 + 747,495a^{11}A^2B^5 + 509,355a^{13}AB^6 \\
& + 135,135a^{15}B^7)t^7}{7!} \\
& + + + +
\end{aligned} \tag{21}$$

This is the direct series solution for Equation (13) approximated by the first eight terms. The (infinite) series converges if B is "small enough". (However, see Section 11.)

Figure (1) shows, in graphical form, the result of substituting the indicated values in nineteen terms of Equation (21) extended. The values for the direct series solution are shown superimposed on a solution by numerical integration.

Finite power series solutions of this type only represent the functions for relatively small values of the independent variable. Eventually the sign of the last term causes the series solution to go above or below the numerical solution.



If the computer data for Equation (20) (nineteen terms) are sorted on B^n and the p^{-n} terms are summed (if $\|p\|$ is large) the first three terms are

$$F(p) = \frac{ap}{(p+A)} + \frac{a^3 B p}{(p+A)(p+3A)} + \frac{3a^5 B^2 p}{(p+A)(p+3A)(p+5A)} + \dots \quad (22)$$

If these terms are evaluated by the inverse transform

$$\begin{aligned} f(t) = & a e^{-At} - \frac{a^3 B}{2A} e^{-At} (1 - e^{-2At}) \\ & + \frac{3}{8} \frac{a^5 B^2}{A^2} e^{-At} (1 - e^{-2At})^2 \\ & - \frac{5}{16} \frac{a^7 B^3}{A^3} e^{-At} (1 - e^{-2At})^3 \\ & + \dots \end{aligned} \quad (23)$$

This is the "summation solution".

Convergence is assured if B is small enough.

To test for uniform convergence by the "M" test

$$\sum_{n=0}^{\infty} M_n = a + \frac{a^3 B}{2A} + \frac{3}{8} \frac{a^5 B^2}{A^2} + \frac{5}{16} \frac{a^7 B^3}{A^3} \quad (24)$$

since all of the exponential coefficients of (24) are less than one for any value of t since

$$\sum_{n=0}^{\infty} M_n = \frac{a}{\sqrt{1 - \frac{a^2 B}{A}}} \quad (25)$$

provided B is small enough.

The series (23) is therefore absolutely and uniformly convergent $t \geq 0$ for

$$0 < \frac{a^2 B}{A} < 1. \quad (26)$$

In this case the terms of Equation (23) may be summed by binomial expansion provided

$$\left| \frac{a^2 B}{A} (1 - e^{-2At}) \right| < 1 \quad (27)$$

giving

$$f(t) = \frac{a e^{-At}}{\sqrt{1 + \frac{a^2 B}{A} (1 - e^{-2At})}}. \quad (28)$$

If the solution of one case is known the result may sometimes be stated in a more general form.

The solution of the equation

$$\frac{df}{dt} + Af + Bf^q = 0 \quad (29)$$

where q is a positive integer is:

$$f(t) = \frac{a e^{-At}}{\left[1 - \frac{a^{(q-1)} B}{A} (e^{-A(q-1)t} - 1) \right]^{\frac{1}{q-1}}} \quad (30)$$

for suitable values of the constants.

5. SOLUTION BY EXPANSION

The Laplace Transform method of solution may easily be adapted to machine calculations using a method which has been designated as the "expansion" method since the key operation is to expand the solution as

$$f = f_1 + f_2 + f_3 + \dots \quad (31)$$

The results appear to match solutions obtained by the summation derivative process of Section 4. In fact the operational root functions for the summation procedure are readily predicted by a rough study using the expansion principle.

As an example in the use of the method consider the equation

$$f' + xf = ge^{-yt} + k(f^2 + ff') \quad (32)$$

with $f(0) = a$. $f(t)$ may only be a bounded function.

The direct transform gives

$$(p + x)F(p) = \frac{g p}{p + y} + ap + k \mathcal{L}(f^2 + ff') \quad (33)$$

and

$$F(p) = \frac{g p}{(p+x)(p+y)} + \frac{a p}{p + x} + \frac{k}{p + x} \mathcal{L}(f^2 + ff') \quad (34)$$

Applying the inverse transform to Equation (34)

$$\begin{aligned} f(t) &= \frac{g(e^{-xt} - e^{-yt})}{y - x} + a e^{-xt} \\ &\quad + k \mathcal{L}^{-1} \left[\frac{1}{p + x} \mathcal{L}(f^2 + ff') \right] \\ &\equiv f_1 + f_2 + f_3 + \dots \end{aligned} \quad (35)$$

where

$$f_1 = g \frac{(e^{-xt} - e^{-yt})}{y - x} + a e^{-xt}. \quad (36)$$

Equation (35) is more easily carried out by changing to Heaviside notation using the definition

$$D \equiv \frac{1}{p + x} \quad (37)$$

and expanding the non-linear terms

$$\begin{aligned} f_1 + f_2 + f_3 + \dots = \\ f_1 + kD \left[f_1^2 + f_1 f_1' \right] \\ + kD \left[f_2^2 + 2f_1 f_2 + f_1' f_2 + f_1 f_2' + f_2 f_1' \right] \\ + kD \left[f_3^2 + 2f_1 f_3 + 2f_2 f_3 \right. \\ \left. + f_1' f_3 + f_3 f_2' + f_1 f_3' + f_2 f_3' + f_3 f_3' \right] \\ + \dots \end{aligned} \quad (38)$$

In Equation (38) each bracket term on the right can be paired with a term on the left

$$f_2 = kD \left[f_1^2 + f_1 f_1' \right] \quad (39)$$

$$\begin{aligned} f_3 = kD \left[f_2^2 + 2f_1 f_2 + f_1' f_2 \right. \\ \left. + f_1 f_2' + f_2 f_1' \right] \end{aligned} \quad (40)$$

$$\begin{aligned} f_4 = kD \left[f_3^2 + 2f_1 f_3 + 2f_2 f_3 \right. \\ \left. + f_1' f_3 + f_3 f_2' + f_1 f_3' + f_2 f_3' + f_4 f_3' \right]. \end{aligned} \quad (41)$$

Since f_1 is known f_2 may be obtained by a "Heaviside shift" which may easily be programmed on a computer. When f_2 is known f_3 may be found. This procedure may be continued to obtain as many terms as required.

The computer program is based on the following operation:

$$\mathcal{L}^{-1} \left[\frac{1}{p+y} \mathcal{L} (e^{xt} t^n) \right] = f(t) \quad (42)$$

where x and y are real and n is a positive integer including zero.

There are two cases:

Case I $y = -x$

$$f(t) = \frac{e^{xt} t^{n+1}}{n+1} \quad (43)$$

Case II $y \neq -x$

$$f(t) = (n!) \left[\frac{e^{-yt}}{\{- (x+y)\}^{(n+1)}} \right. \\ \frac{-e^{xt} t^n}{\{- (x+y)\}^n n!} \\ \frac{-e^{xt} t^{n-1}}{\{- (x+y)\}^{(n-1)} (n-1)!} \\ \text{---} \\ \left. \frac{-e^{xt} t^0}{\{- (x+y)\}^{(n+1)} (0)!} \right] \quad (44)$$

While the machine program handles only real values of x and y the results may be converted into sinusoidal quantities quite readily.

6. THE PENDULUM

The equation of motion for an idealized pendulum is

$$\frac{d^2 \theta}{dt^2} = -k \sin \theta \quad (45)$$

where k is the ratio of the gravitational constant to the length of the pendulum. $\theta(t)$ is the angle of displacement in radians.

The solution for the period of oscillation is well known¹⁴ in this case and the expansion method would generate secular terms since $\sin \theta$ would be approximated by the first few terms in θ .

The derivative method also generates secular terms if the data is sorted on k before summation. Knowing the period¹⁴ one may approximate the rotational velocity as

$$\omega_1 = \frac{\sqrt{k}}{1 + \left(\frac{1}{2}\right)^2 \beta^2 + \left(\frac{1.3}{2.4}\right)^2 \beta^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 \beta^6 + \dots} \quad (46)$$

where $\beta^2 = \frac{b^2}{4k}$ and $b = \theta'(0)$.

If data in p is then summed on the basis of the odd harmonics of ω_1 for $f(0) = 0$, $f'(0) = b$

$$\begin{aligned} F(p) = & \frac{bp}{(p^2 + \omega_1^2)} - \left[\frac{b^3}{8} + \frac{3b^5}{512k} + \frac{3b^7}{4096k^2} + \dots \right] \frac{p}{(p^2 + \omega_1^2)(p^2 + 9\omega_1^2)} \\ & + \left[\frac{3b^5}{32} + \frac{15b^7}{2048k} + \frac{513b^9}{262,144k^2} \right. \\ & \left. + \frac{81b^{11}}{1,048,576k^3} + \frac{81b^{13}}{16,777,216k^4} + \dots \right] \frac{p}{(p^2 + \omega_1^2)(p^2 + 9\omega_1^2)(p^2 + 25\omega_1^2)} \\ & + \dots \end{aligned} \quad (47)$$

The transforms may easily be evaluated to obtain the approximation

$$\begin{aligned} \theta(t) = & \frac{b}{\omega_1} \sin \omega_1 t - \left[\frac{b^3}{8} + \frac{3b^5}{512k} + \frac{3b^7}{4096k^2} + + \right] \frac{\sin^3 \omega_1 t}{6\omega_1^3} \\ & + \left[\frac{3b^5}{32} + \frac{15b^7}{2048k} + \frac{513b^9}{262,144k^2} + \frac{81b^{11}}{1,048,576k^3} \right. \\ & \left. + \frac{81b^{13}}{16,777,216k^4} + + + \right] \frac{\sin^5 \omega_1 t}{150\omega_1^5} \end{aligned} \quad (48)$$

The maximum value of $b = f'(0)$ for which ω_1 is valid is $2\sqrt{k}$. If this value is used for the series forming the coefficients they appear to converge very rapidly and it is assumed that they do converge.

If $\sin \omega_1 t$ is replaced by its maximum value one may then examine the first three terms of the alternating series. Again they appear to be absolutely convergent and the assumption is made that the result is absolutely and uniformly convergent and is therefore a solution for the original differential equation for the case where $f(0) = 0$ and $f'(0) = b$ in radians per second.

Figures (2) through (5) show analytic results compared with results obtained by numerical integration for four cases.

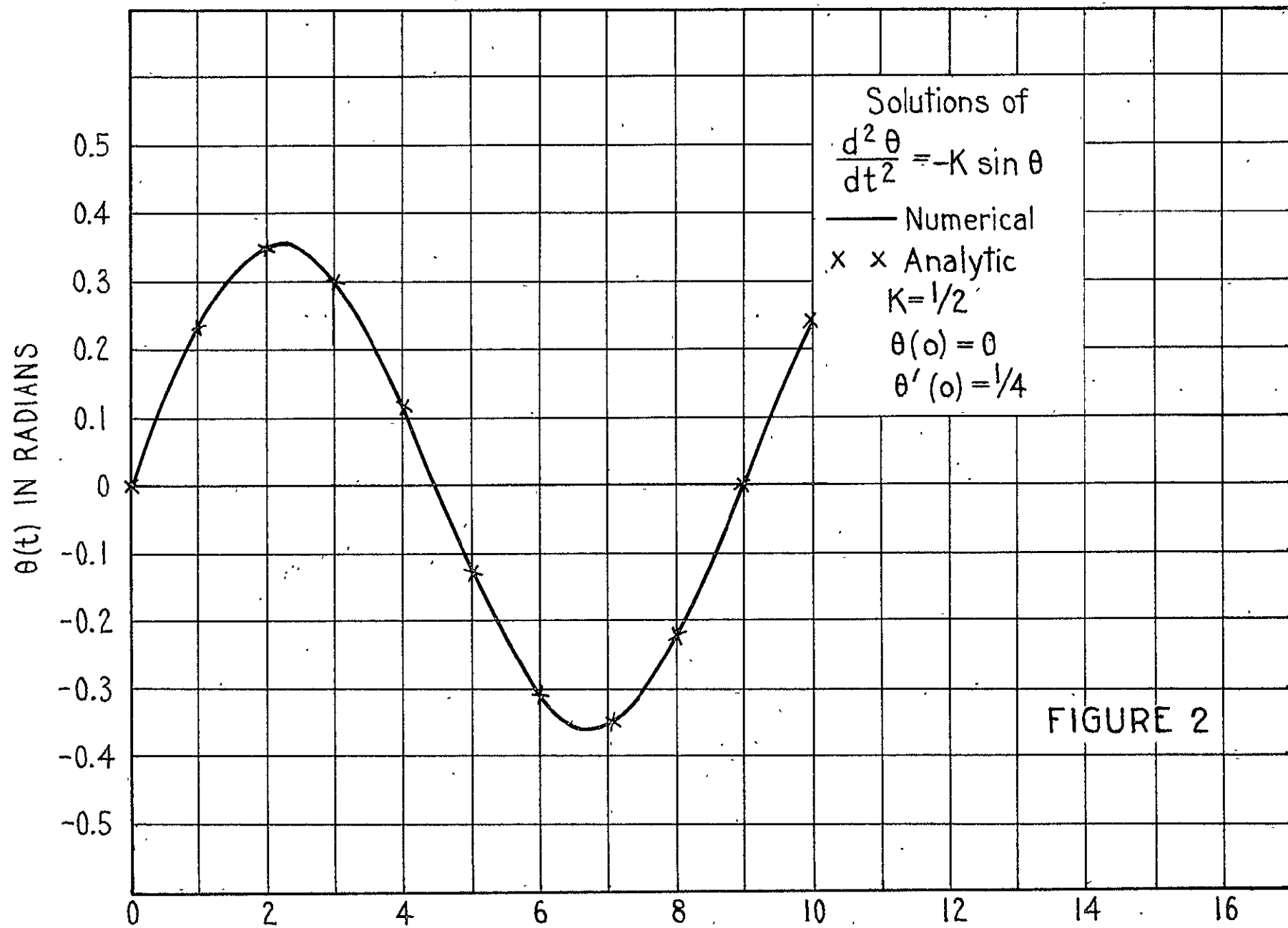


FIGURE 2

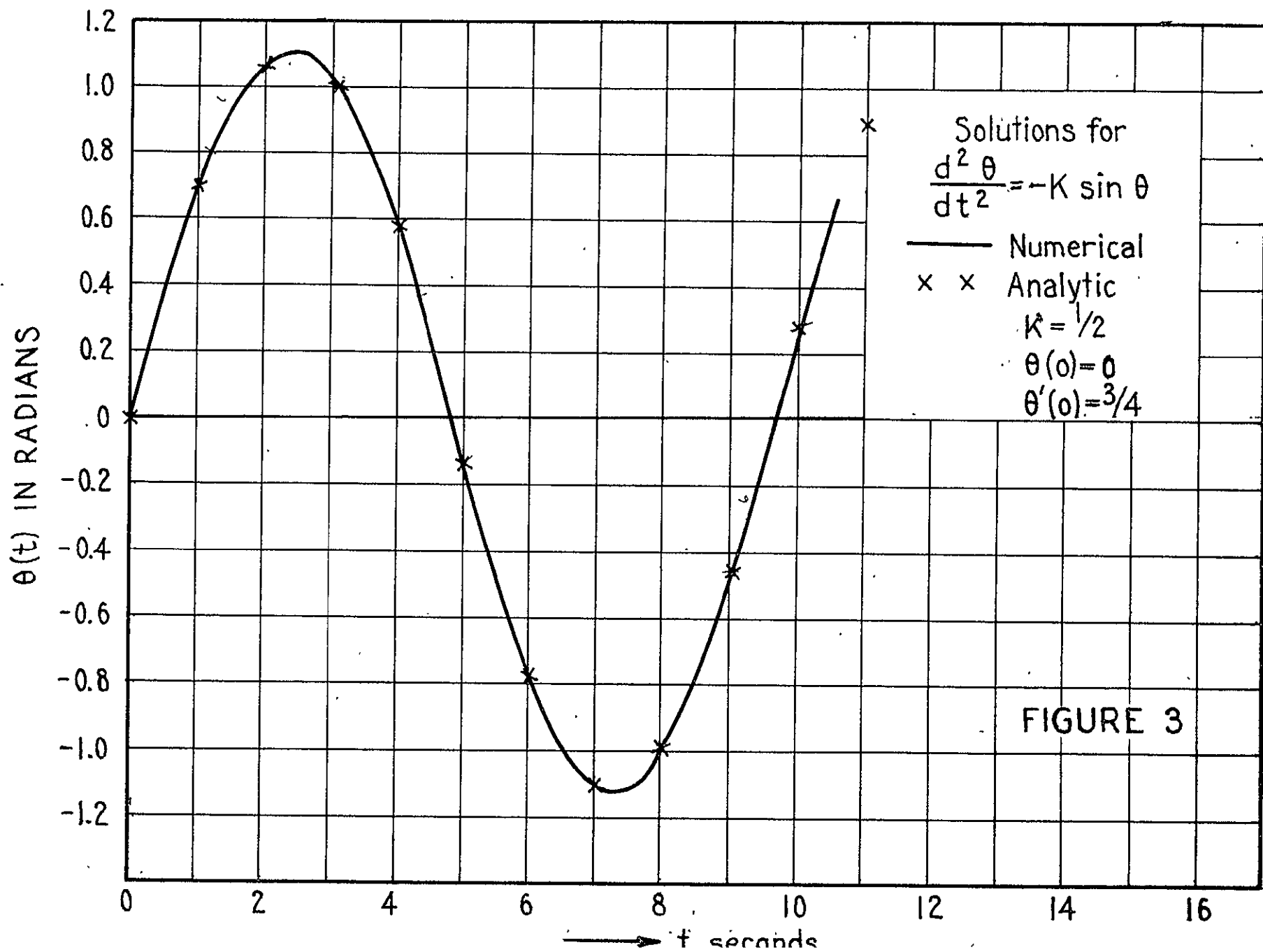


FIGURE 3

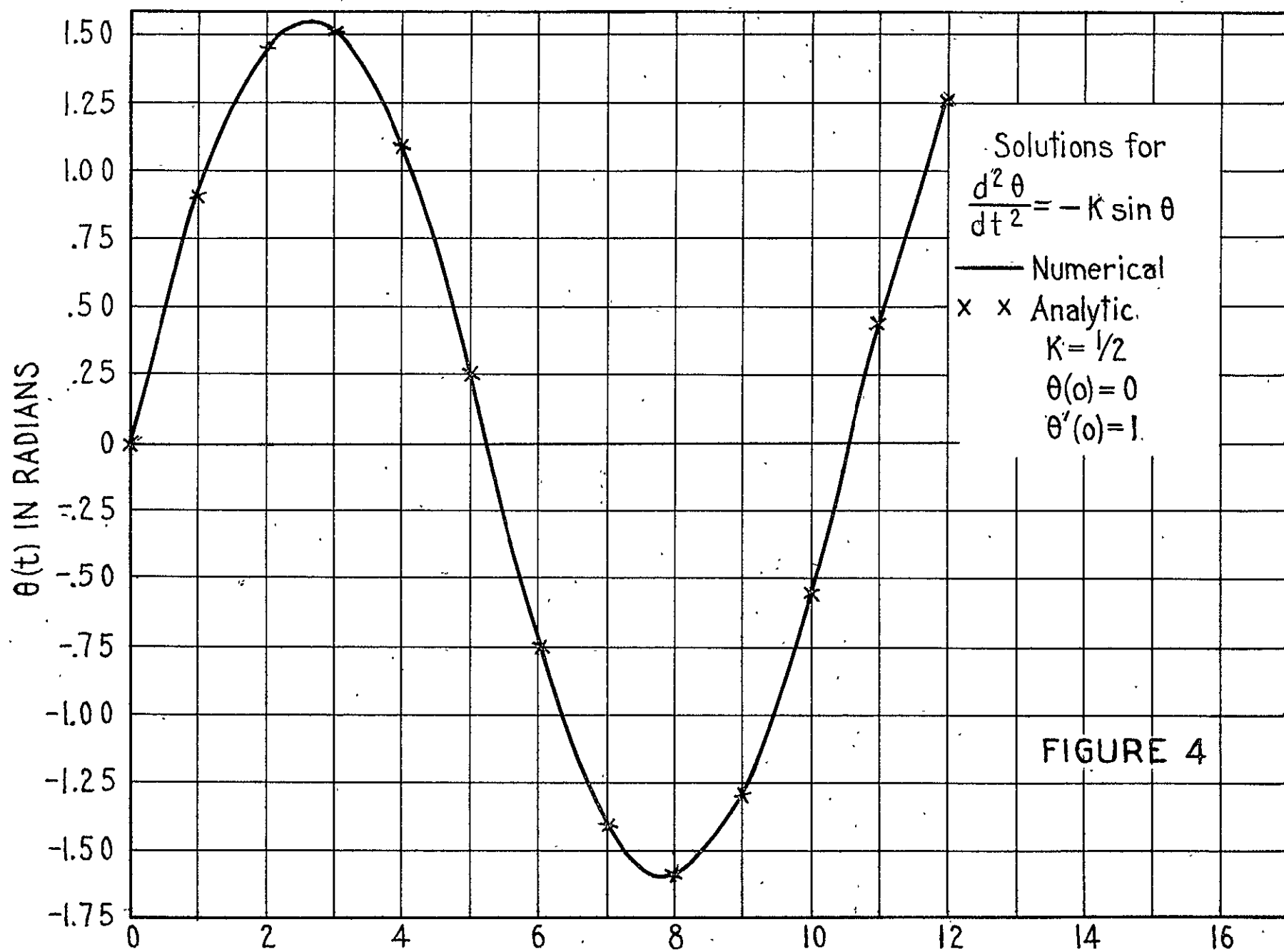
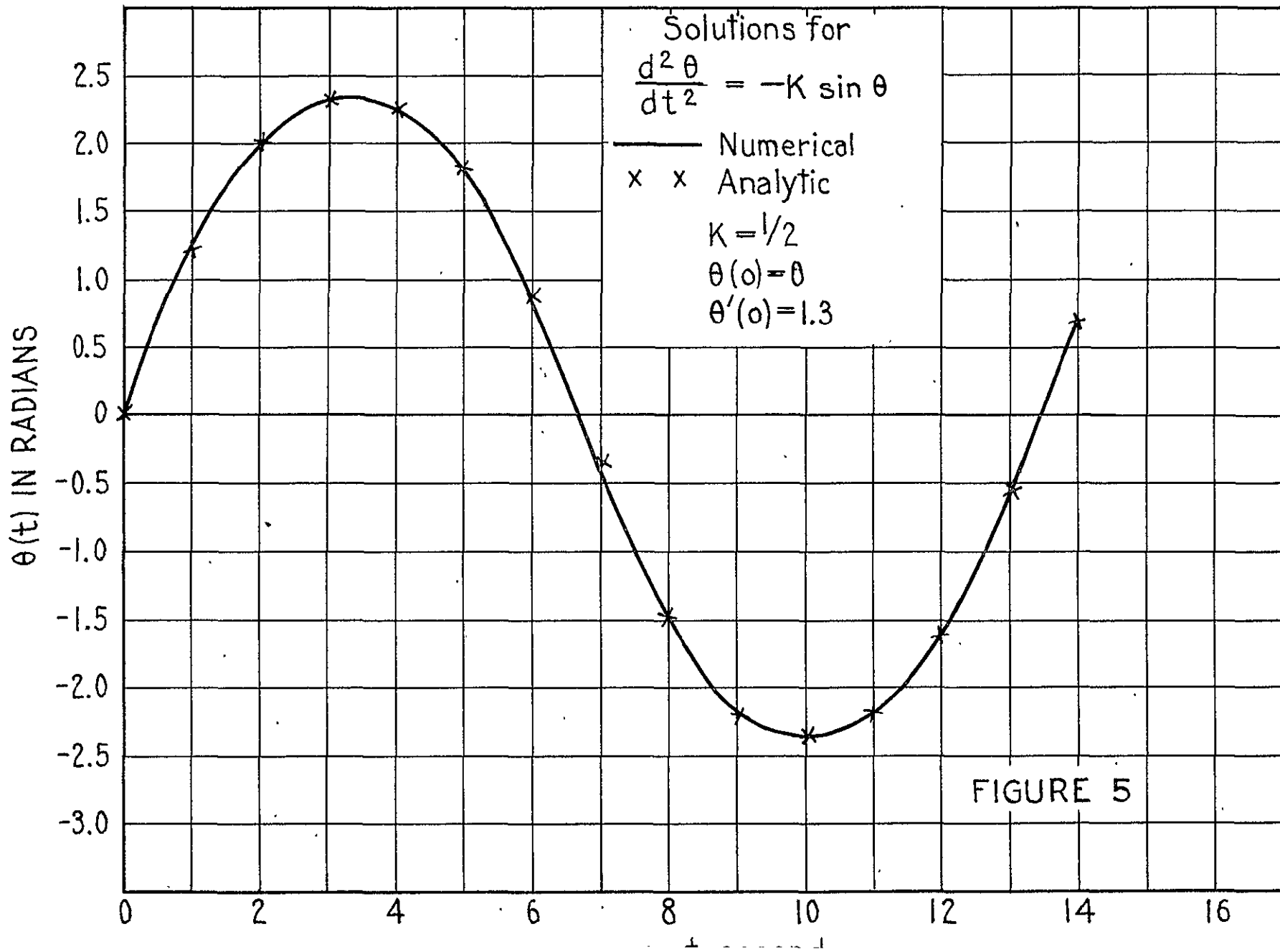


FIGURE 4



7. SECOND ORDER EQUATION

One of the most interesting equations under study was the second order equation

$$f'' + (x+y) f' + xyf = kf^2. \quad (49)$$

Here x and y are real and positive while k is a small positive or negative real number.

In addition

$$\pm m x \neq n y \quad (50)$$

where m and n are any positive integers. If this inequality does not hold multiple roots will appear causing secular terms in the final result.

Evidently x and y must be irrational numbers.

In the following operational transforms the initial conditions will be abbreviated as

$$f(0) = a, f'(0) = b. \quad (51)$$

The first three terms of the complete solution by the derivative operational method are

$F(p) =$

$$\begin{aligned} & \frac{p(ap+b+ax+ay)}{(p+x)(p+y)} + \\ & + kp \frac{(p^2a^2+2pab+4abx+2b^2+3pa^2x+2a^2x^2+3pa^2y+4aby+6a^2xy+2a^2y^2)}{(p+x)(p+y)(p+2x)(p+x+y)(p+2y)} \\ & + k^2 p \left[\frac{2p^4a^3+16p^3a^3x+10p^3a^2b+102p^2a^3xy+62p^2a^2bx+62p^2a^2by+20p^2ab^2+}{(p+x)(p+y)(p+2x)(p+x+y)(p+2y)(p+3x)(p+2x+y)(p+x+2y)(p+3y)} \right. \\ & \quad 206pa^3x^2y+120pa^2bx^2+270pa^2bxy+84pab^2x+84pab^2y+20pb^3+ \\ & \quad 132a^3x^3y+72a^2bx^3+252a^2bx^2y+72ab^2x^2+144ab^2xy+24b^3x+ \\ & \quad 24b^3y+16p^3a^3y+46p^2a^3x^2+56pa^3x^3+46p^2a^3y^2+24a^3x^4+206pa^3xy^2+ \\ & \quad 120pa^2by^2+56pa^3y^3+204a^3x^2y^2+252a^2by^2x+72ab^2y^2+132a^3xy^3+ \\ & \quad \left. 72a^2by^3+24a^3y^4 \right] \end{aligned} \quad (52)$$

If this equation is evaluated it represents three terms of a formal solution of Equation (49). Each term multiplied by a given power of k is made up of negative exponentials added together. If x and y are irrational numbers there will be no repeated roots. If it is assumed that the infinite series of such terms behave in the same way some finite constant M_m may always be chosen to replace each k multiplier and we have

$$\sum_{n=0}^{\infty} k^n M_n > f(t). \quad (53)$$

If these M_n follow a normal pattern they eventually decrease with increasing n so that all M_n may be replaced by the maximum value M .

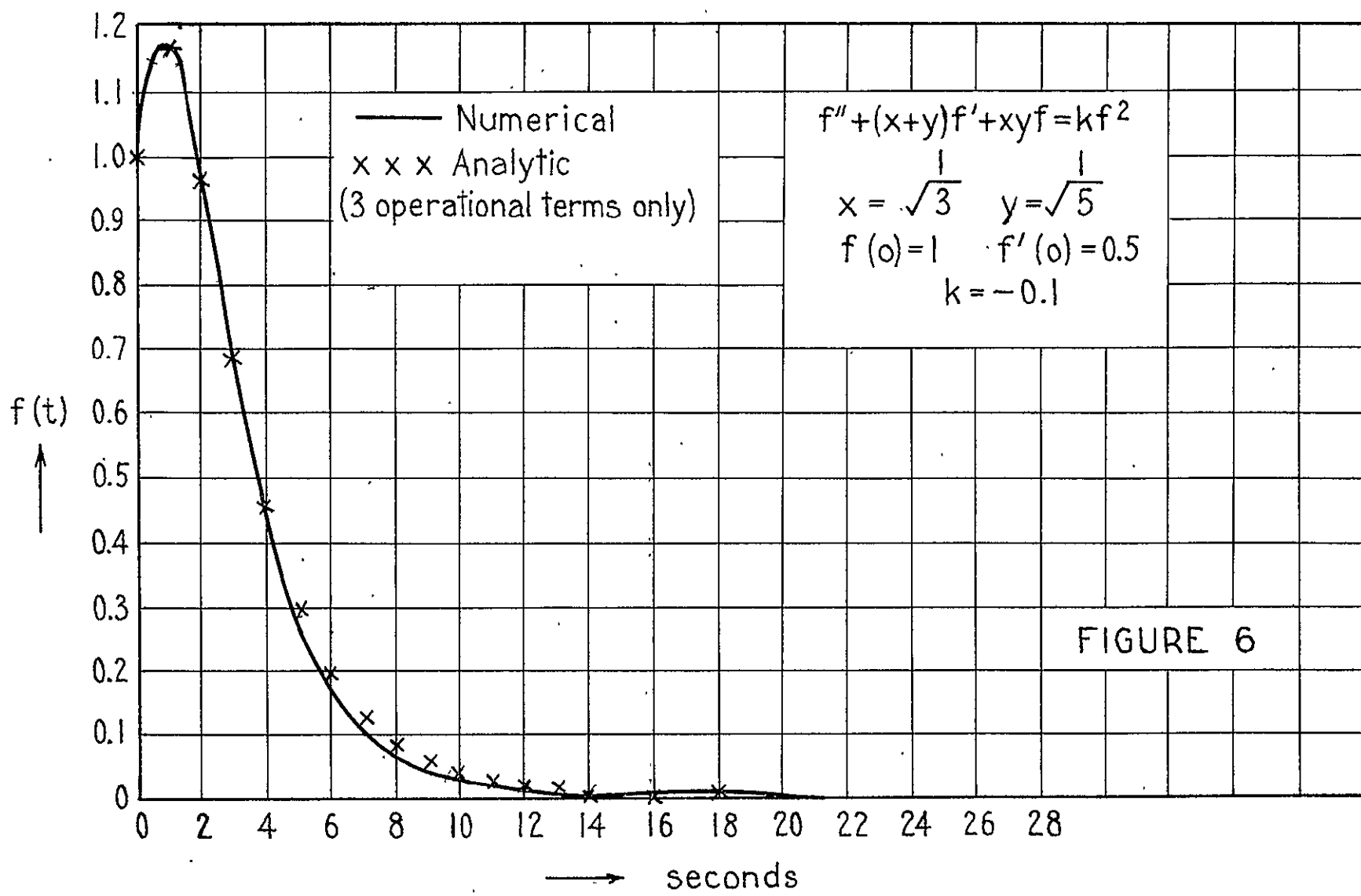
$$M \sum_{n=0}^{\infty} k^n > f(t) \quad (54)$$

Evidently the result, under these assumptions, converges absolutely and uniformly for $t \geq 0$ when k is sufficiently small.

Fortunately it is not necessary to rely on so many assumptions since Poincaré's theorem assures convergence for sufficiently small values of k in any case.

Now Equation (52) is the solution of Equation (49) in operational form. The operational solution is sometimes as interesting as the final evaluation for $f(t)$. In this case the initial and final value theorems applied to the individual terms show that the final value of $f(t)$ is zero for all three terms and the initial value is zero except for the first term where $f(0) = a$.

A computer program evaluated Equation (52) directly for given values of x , y , $f(0)$, $f'(0)$ and t . The result is shown as Figure 6.



Memory space in the computer is always at a premium with these solutions. While Equation (52) contained both initial conditions (a and b) the next term would have required more memory than available. One may obtain more terms (k^n) by sorting data either $f(0) = a = 0$ or $f'(0) = b = 0$.

Equation (55) shows the operational solution for $f'(0) = b = 0$ and $f(0) = a$

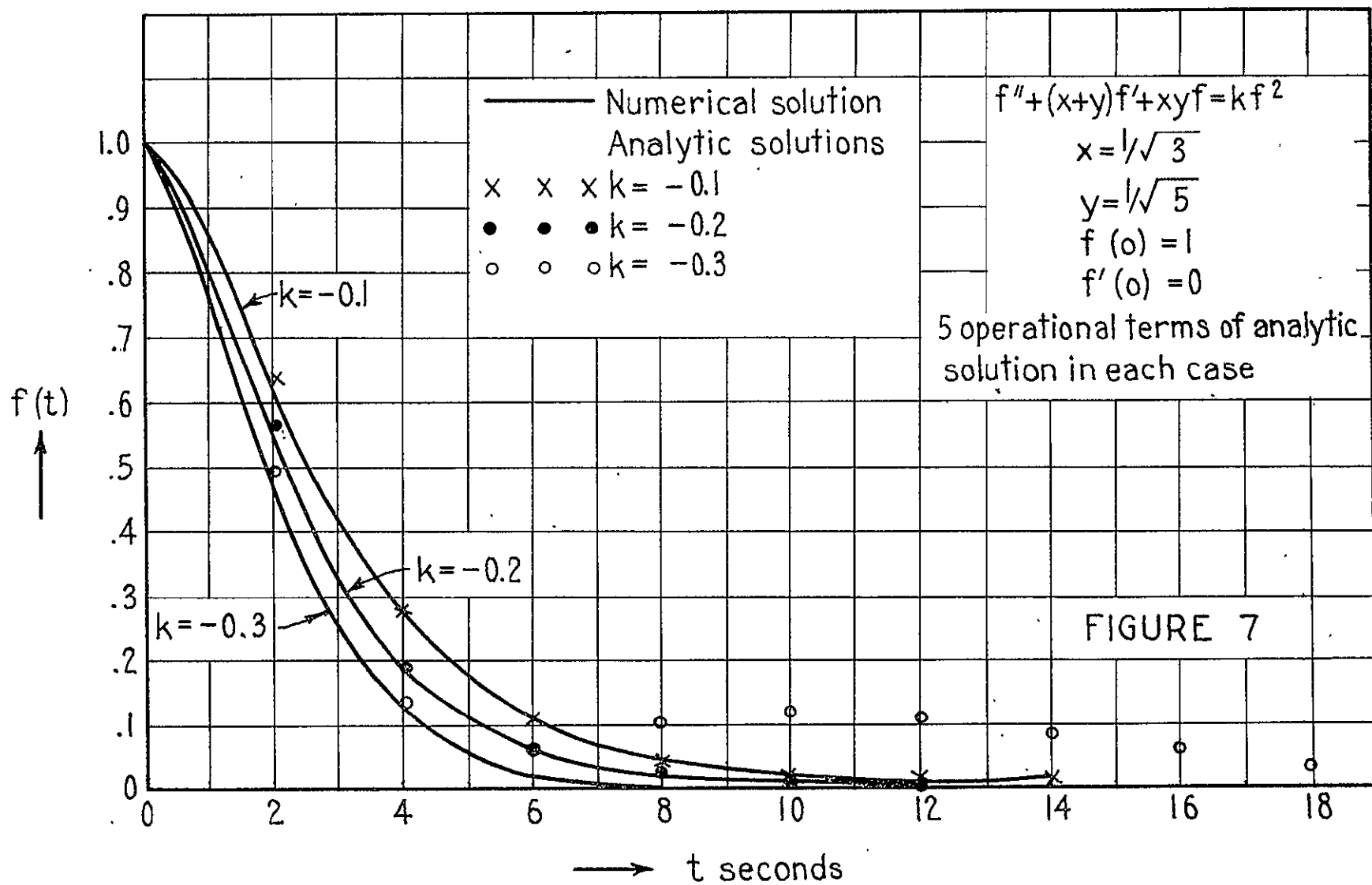
$$\begin{aligned}
 F(p) = & \frac{a(p^2 + xp + yp)}{(p+x)(p+y)} + \frac{Ka^2(p^3 + 3xp^2 + 2x^2p + 3yp^2 + 6xyp + 2y^2p)}{(p+x)(p+y)(p+2x)(p+2y)(p+x+y)} \\
 & + \frac{K^2a^3 \left[\begin{aligned} & 2p^5 + 16xp^4 + 46x^2p^3 + 102xyp^3 + 206x^2yp^2 + 206xy^2p^2 + 132x^3yp \\ & + 204x^2y^2p + 16yp^4 + 46y^2p^3 + 56x^3p^2 + 56y^3p^2 + 24x^4p \\ & + 132xy^3p + 24y^4p \end{aligned} \right]}{(p+x)(p+y)(p+2x)(p+x+y)(p+2y)(p+3x)(p+2x+y)(p+x+2y)(p+3y)} \\
 & + \frac{K^3a^4 \left[\begin{aligned} & 10p^8 + 162xp^7 + 162yp^7 + 1090x^2p^6 + 3942x^3p^5 + 13,466x^2yp^5 \\ & + 13,466xy^2p^5 + 39,922x^3yp^4 + 63,764x^2y^2p^4 + 144,136x^3y^2p^3 \\ & + 144,136x^2y^3p^3 + 154,944x^4y^2p^2 + 219,744x^3y^3p^2 + 63,360x^5y^2p \\ & + 118,656x^4y^3p + 2330xyp^6 + 1090y^2p^6 + 3942y^3p^5 + 8260x^4p^4 \\ & + 63,856x^4yp^3 + 39,922xy^3p^4 + 10,008x^5p^3 + 52,152x^5yp^2 \\ & + 8260y^4p^4 + 6480x^6p^2 + 16,992x^6yp + 63,856xy^4p^3 + 10,008y^5p^3 \\ & + 1728x^7p + 154,944x^2y^4p^2 + 52,152xy^5p^2 + 6480y^6p^2 + 118,656x^3y^4p \\ & + 63,360x^2y^5p + 16,992xy^6p + 1728y^7p \end{aligned} \right]}{(p+x)(p+y)(p+2x)(p+x+y)(p+2y)(p+3x)(p+2x+y)(p+x+2y)(p+3y)(p+4x)(p+3x+y)(p+2x+2y)(p+x+3y)(p+4y)}
 \end{aligned}$$

$$\begin{aligned}
& 80p^{12} + 2336xp^{11} + 2336yp^{11} + 30,288x^2p^{10} + 63,176xyp^{10} \\
& + 749,840x^2yp^9 + 749,840xy^2p^9 + 5,137,232x^3yp^8 + 8,029,904x^2y^2p^8 \\
& + 5,137,232xy^3p^8 + 22,467,296x^4yp^7 + 48,758,176x^3y^2p^7 \\
& + 48,758,176x^2y^3p^7 + 22,467,296xy^4p^7 + 65,443,304x^5yp^6 \\
& + 184,749,440x^4y^2p^6 + 256,459,560x^3y^3p^6 + 184,749,440x^2y^4p^6 \\
& + 128,382,000x^6yp^5 + 452,361,232x^5y^2p^5 + 816,660,896x^4y^3p^5 \\
& + 816,660,896x^3y^4p^5 + 452,361,232x^2y^5p^5 + 714,697,184x^6y^2p^4 \\
& + 1,610,209,192x^5y^3p^4 + 2,093,793,712x^4y^4p^4 + 1,610,209,192x^3y^5p^4 \\
& + 714,697,184x^2y^6p^4 + 701,644,736x^7y^2p^3 + 1,917,317,088x^6y^3p^3 \\
& + 3,110,628,512x^5y^4p^3 + 3,110,628,512x^4y^5p^3 + 1,917,317,088x^3y^6p^3 \\
& + 388,007,040x^8y^2p^2 + 1,259,061,216x^7y^3p^2 + 2,477,021,376x^6y^4p^2 \\
& + 3,089,805,696x^5y^5p^2 + 2,477,021,376x^4y^6p^2 + 92,044,800x^9y^2p \\
& + 348,842,880x^8y^3p + 814,798,080x^7y^4p + 1,231,879,680x^6y^5p \\
& + 1,231,879,680x^5y^6p + 30,288y^2p^{10} + 229,920x^3p^9 + 1,134,000x^4p^8 \\
& + 229,920y^3p^9 + 3,810,528x^5p^7 + 1,134,000y^4p^8 + 8,888,944x^6p^6 \\
& + 65,443,304xy^5p^6 + 3,810,528y^5p^7 + 8,888,944y^6p^6 + 167,238,528x^7yp^4 \\
& + 14,374,240x^7p^5 + 128,382,000xy^6p^5 + 138,276,288x^8yp^3 \\
& + 15,769,920x^8p^4 + 14,374,240y^7p^5 + 65,473,920x^9yp^2 + 11,166,336x^9p^3 \\
& + 167,238,528xy^7p^4 + 15,769,920y^8p^4 + 13,478,400x^{10}yp \\
& + 4,589,568x^{10}p^2 + 701,644,736x^2y^7p^3 + 138,276,288xy^8p^3 \\
& + 11,166,336y^9p^3 + 829,440x^{11}p + 1,259,061,216x^3y^7p^2 \\
& + 388,007,040x^2y^8p^2 + 65,473,920xy^9p^2 + 4,589,568y^{10}p^2 \\
& + 814,798,080x^4y^7p + 348,842,880x^3y^8p + 92,044,800x^2y^9p \\
& + 13,478,400xy^{10}p + 829,440y^{11}p \\
& + K a^5 \left[\begin{aligned}
& (p+x)(p+y)(p+2x)(p+x+y)(p+2y)(p+3x)(p+2x+y)(p+x+2y) \\
& (p+3y)(p+4x)(p+3x+y)(p+2x+2y)(p+x+3y)(p+4y)(p+5x) \\
& (p+4x+y)(p+3x+2y)(p+2x+3y)(p+x+4y)(p+5y)
\end{aligned} \right]
\end{aligned}$$

Equation (56) is the operational solution for $f(0) = a = 0$ and $f'(0) = b$.

$$\begin{aligned}
 F(p) = & \frac{bp}{(p+x)(p+y)} + \frac{2Kb^2p}{(p+x)(p+y)(p+2x)(p+x+y)(p+2y)} \\
 & + \frac{K^2b^3(20p^2 + 24px + 24py)}{(p+x)(p+y)(p+2x)(p+x+y)(p+2y)(p+3x)(p+2x+y)(p+x+2y)(p+3y)} \\
 & + \frac{K^3b^4 \left[600p^4 + 2760(xp^3 + yp^3) + 3888(x^2p^2 + y^2p^2) + 8976xyp^2 \right. \\
 & \quad \left. + 6624(x^2yp + xy^2p) + 1728(x^3p + y^3p) \right]}{(p+x)(p+y)(p+2x)(p+x+y)(p+2y)(p+3x)(p+2x+y)(p+x+2y)(p+3y)(p+4x)(p+3x+y)(p+2x+2y)(p+x+3y)(p+4y)} \\
 & + \frac{K^4b^5 \left[39,600p^7 + 443,520(xp^6 + yp^6) + 1,972,080(x^2p^5 + y^2p^5) \right. \\
 & \quad + 4,274,160xyp^5 + 4,457,376(x^3p^4 + y^3p^4) + 15,682,128(x^2yp^4 + xy^2p^4) \\
 & \quad + 5,405,184(x^4p^3 + y^4p^3) + 27,402,336(x^3yp^3 + xy^3p^3) \\
 & \quad + 44,428,704x^2y^2p^3 + 3,345,408(x^5p^2 + y^5p^2) \\
 & \quad + 22,809,600(x^4yp^2 + xy^4p^2) + 53,211,840(x^3y^2p^2 + x^2y^3p^2) \\
 & \quad + 7,257,600(x^5yp + xy^5p) + 22,752,000(x^4y^2p + x^2y^4p) \\
 & \quad \left. + 32,647,680(x^3y^3p) + 829,440(x^6p + y^6p) \right]}{(p+x)(p+y)(p+2x)(p+x+y)(p+2y)(p+3x)(p+2x+y)(p+x+2y)(p+3y)(p+4x)(p+3x+y)(p+2x+2y)(p+x+3y)(p+4y)(p+5x)(p+4x+y)(p+3x+2y)(p+2x+3y)(p+x+4y)(p+5y)}
 \end{aligned}
 \tag{56}$$

Figure (7) shows the results for the analytic solution compared with the results of numerical integration for k negative. Evidently the magnitude of $k = -0.3$ is "too large" for convergence or more than 5 operational terms are required.



The actual evaluation of Equation (49) as a time function is more quickly and more easily carried out by the expansion method, but in this case any numerical check solution must be programmed on the computer.

As an example take the case where $f(0) = 0$ and $f'(0) = b$, which is the evaluation of Equation (56). Notice particularly that the constant "a" is not $f(0)$ as usual but a newly defined constant.

Equation (49) may be written in Heaviside notation with one boundary condition as

$$\left[p^2 + (x+y)p + xy \right] .f = bp + kf^2 \quad (57)$$

where $f(0) = 0$ and $f'(0) = b$

$$f(t) = \frac{bp}{(p+x)(p+y)} + \frac{kf^2}{(p+x)(p+y)} \quad (58)$$

$$\frac{1}{(p+x)(p+y)} = \frac{a}{(p+x)} + \frac{c}{p+y} \quad (59)$$

where

$$a \equiv \frac{1}{y-x} \quad \text{and} \quad c \equiv \frac{1}{x-y}$$

$$f(t) = a b e^{-xt} + b c e^{-yt} + \left[\frac{a}{p+x} + \frac{c}{p+y} \right] k f^2 \quad (60)$$

$$f \equiv f_1 + f_2 + f_3 + f_4 + + + \quad (61)$$

and

$$\begin{aligned} f^2 &= f_1^2 \\ &+ (f_2^2 + 2f_1f_2) \\ &+ (f_3^2 + 2f_1f_3 + 2f_2f_3) \\ &+ (f_4^2 + 2f_1f_4 + 2f_2f_4 + 2f_3f_4) \\ &+ + + \end{aligned} \quad (62)$$

$$f(t) = f_1 + f_2 + f_3 + f_4 + + \text{ and}$$

$$D \equiv \left[\frac{a}{p+x} + \frac{c}{p+y} \right] \quad (63)$$

$$f_1 = ab e^{-xt} + bc e^{-yt} \quad (64)$$

$$f_2 = Dk (f_1^2) \quad (65)$$

$$f_3 = Dk (f_2^2 + 2f_1 f_2) \quad (66)$$

$$f_4 = Dk (f_3^2 + 2f_1 f_3 + 2f_2 f_3). \quad (67)$$

In carrying these operations out with the computer the two terms of f_1 are set up as shown in Equation (64). A multiply program then forms f_1^2 . All terms are then given a common denominator so they may be packed into the smallest number of terms. The next step is to multiply all terms by k . Each of the operators of Equation (63) is then applied individually to kf_1^2 . A common denominator is then found so that the terms may be packed. The result is then read out in decimal and punched as a hexadecimal tape for future operations.

The results are carried in the machine as algebraic quantities and exponentials so there is no need for laborious hand calculations.

In this particular case no secular terms appear since it is assumed that the inequality of Equation (50) is satisfied.

The approximation carried out through the k^3 term only is given by

$$f(t) = f_1 + f_2 + f_3 + f_4 + \dots$$

Where

$$f_1 = ab e^{-xt} + bc e^{-yt}$$

$$f_2 = kb^2 \left[\begin{aligned} & -\frac{a^3 e^{-xt}(e^{-xt}-1)}{x} - \frac{2a^2 c e^{-xt}(e^{-yt}-1)}{y} + \frac{ac^2 e^{-xt}(e^{(x-2y)t}-1)}{x-2y} \\ & - \frac{a^2 c e^{-yt}(e^{(y-2x)t}-1)}{2x-y} - \frac{2ac^2 e^{-yt}(e^{-xt}-1)}{x} - \frac{c^3 e^{-yt}(e^{-yt}-1)}{y} \end{aligned} \right]$$

$$f_3 = k^2 b^3 \left[\begin{aligned} & + \frac{a^5 e^{-xt}(e^{-2xt}-1)}{x^2} - \frac{2a^5 e^{-xt}(e^{-xt}-1)}{x^2} + \frac{4a^4 c e^{-xt}(e^{-(x+y)t}-1)}{y(x+y)} \\ & - \frac{4a^4 c e^{-xt}(e^{-xt}-1)}{xy} - \frac{a^3 c^2 e^{-xt}(e^{-2yt}-1)}{y(x-2y)} + \frac{2a^3 c^2 e^{-xt}(e^{-xt}-1)}{x(x-2y)} \\ & + \frac{a^4 c e^{-xt}(e^{-2xt}-1)}{x(2x-y)} - \frac{2a^4 c e^{-xt}(e^{-yt}-1)}{y(2x-y)} + \frac{4a^3 c^2 e^{-xt}(e^{-(x+y)t}-1)}{x(x+y)} \\ & - \frac{4a^3 c^2 e^{-xt}(e^{-yt}-1)}{xy} + \frac{a^2 c^3 e^{-xt}(e^{-2yt}-1)}{y^2} - \frac{2a^2 c^3 e^{-xt}(e^{-yt}-1)}{y^2} \\ & + \frac{2a^4 c e^{-xt}(e^{-(x+y)t}-1)}{x(x+y)} - \frac{2a^4 c e^{-xt}(e^{-yt}-1)}{xy} + \frac{2a^3 c^2 e^{-xt}(e^{-2yt}-1)}{y^2} \\ & - \frac{4a^3 c^2 e^{-xt}(e^{-yt}-1)}{y^2} + \frac{2a^2 c^3 e^{-xt}(e^{(x-3y)t}-1)}{(x-2y)(x-3y)} + \frac{2a^2 c^3 e^{-xt}(e^{-yt}-1)}{y(x-2y)} \\ & + \frac{2a^3 c^2 e^{-xt}(e^{-(x+y)t}-1)}{(2x-y)(x+y)} + \frac{2a^3 c^2 e^{-xt}(e^{(x-2y)t}-1)}{(2x-y)(x-2y)} \\ & + \frac{2a^2 c^3 e^{-xt}(e^{-2yt}-1)}{xy} + \frac{4a^2 c^3 e^{-xt}(e^{(x-2y)t}-1)}{x(x-2y)} \\ & - \frac{2ac^4 e^{-xt}(e^{(x-3y)t}-1)}{y(x-3y)} + \frac{2ac^4 e^{-xt}(e^{(x-2y)t}-1)}{y(x-2y)} \end{aligned} \right]$$

Cont.

f₃ Continued

$$\begin{aligned}
& + k^2 b^3 \left[\begin{aligned}
& + \frac{2a^4 c e^{-yt} (e^{(y-3x)t} - 1)}{x(3x-y)} - \frac{2a^4 c e^{-yt} (e^{(y-2x)t} - 1)}{x(2x-y)} \\
& + \frac{2a^3 c^2 e^{-yt} (e^{-2xt} - 1)}{xy} - \frac{4a^3 c^2 e^{-yt} (e^{(y-2x)t} - 1)}{y(2x-y)} \\
& - \frac{2a^2 c^3 e^{-yt} (e^{-(x+y)t} - 1)}{(x-2y)(x+y)} + \frac{2a^2 c^3 e^{-yt} (e^{(y-2x)t} - 1)}{(2x-y)(x-2y)} \\
& + \frac{2a^3 c^2 e^{-yt} (e^{(y-3x)t} - 1)}{(3x-y)(2x-y)} - \frac{2a^3 c^2 e^{-yt} (e^{-xt} - 1)}{x(2x-y)} + \frac{2a^2 c^3 e^{-yt} (e^{-2xt} - 1)}{x^2} \\
& - \frac{4a^2 c^3 e^{-yt} (e^{-xt} - 1)}{x^2} + \frac{2ac^4 e^{-yt} (e^{-(x+y)t} - 1)}{y(x+y)} - \frac{2ac^4 e^{-yt} (e^{-xt} - 1)}{xy} \\
& + \frac{a^3 c^2 e^{-yt} (e^{-2xt} - 1)}{x^2} - \frac{2a^3 c^2 e^{-yt} (e^{-xt} - 1)}{x^2} + \frac{4a^2 c^3 e^{-yt} (e^{-(x+y)t} - 1)}{y(x+y)} \\
& - \frac{4a^2 c^3 e^{-yt} (e^{-xt} - 1)}{xy} - \frac{ac^4 e^{-yt} (e^{-2yt} - 1)}{y(x-2y)} + \frac{2ac^4 e^{-yt} (e^{-xt} - 1)}{x(x-2y)} \\
& + \frac{a^2 c^3 e^{-yt} (e^{-2xt} - 1)}{x(2x-y)} - \frac{2a^2 c^3 e^{-yt} (e^{-yt} - 1)}{y(2x-y)} + \frac{4ac^4 e^{-yt} (e^{-(x+y)t} - 1)}{x(x+y)} \\
& - \frac{4ac^4 e^{-yt} (e^{-yt} - 1)}{xy} + \frac{c^5 e^{-yt} (e^{-2yt} - 1)}{y^2} - \frac{2c^5 e^{-yt} (e^{-yt} - 1)}{y^2}
\end{aligned} \right] \\
& + k^3 b^4 \left[\begin{aligned}
& - \frac{a^7 e^{-xt} (e^{-3xt} - 1)}{3x^3} + \frac{a^7 e^{-xt} (e^{-2xt} - 1)}{x^3} - \frac{4a^6 c e^{-xt} (e^{-(2x+y)t} - 1)}{xy(2x+y)} \\
& + \frac{2a^6 c e^{-xt} (e^{-2xt} - 1)}{x^2 y} + \frac{2a^5 c^2 e^{-xt} (e^{-(x+2y)t} - 1)}{x(x-2y)(x+2y)} - \frac{a^5 c^2 e^{-xt} (e^{-2xt} - 1)}{x^2(x-2y)} \\
& - \frac{2a^6 c e^{-xt} (e^{-3xt} - 1)}{3x^2(2x-y)} + \frac{2a^6 c e^{-xt} (e^{-(x+y)t} - 1)}{x(2x-y)(x+y)} \\
& - \frac{4a^5 c^2 e^{-xt} (e^{-(2x+y)t} - 1)}{x^2(2x+y)} + \frac{8a^5 c^2 e^{-xt} (e^{-(x+y)t} - 1)}{x^2(x+y)} \\
& - \frac{10a^4 c^3 e^{-xt} (e^{-(x+2y)t} - 1)}{xy(x+2y)} + \frac{10a^4 c^3 e^{-xt} (e^{-(x+y)t} - 1)}{xy(x+y)}
\end{aligned} \right]
\end{aligned}$$

Cont.

f₃ Continued

$$\begin{aligned}
& + k^3 b^4 \left[\begin{aligned}
& - \frac{a^7 e^{-xt} (e^{-xt} - 1)}{x^3} + \frac{4a^6 c e^{-xt} (e^{-(x+y)t} - 1)}{xy(x+y)} - \frac{4a^6 c e^{-xt} (e^{-xt} - 1)}{x^2 y} \\
& - \frac{a^5 c^2 e^{-xt} (e^{-2yt} - 1)}{xy(x-2y)} + \frac{2a^5 c^2 e^{-xt} (e^{-xt} - 1)}{x^2 (x-2y)} + \frac{a^6 c e^{-xt} (e^{-2xt} - 1)}{x^2 (2x-y)} \\
& - \frac{2a^6 c e^{-xt} (e^{-yt} - 1)}{xy(2x-y)} - \frac{4a^5 c^2 e^{-xt} (e^{-yt} - 1)}{x^2 y} + \frac{5a^4 c^3 e^{-xt} (e^{-2yt} - 1)}{xy^2} \\
& - \frac{10a^4 c^3 e^{-xt} (e^{-yt} - 1)}{xy^2} - \frac{4a^5 c^2 e^{-xt} (e^{-(x+2y)t} - 1)}{y^2 (x+2y)} \\
& + \frac{8a^5 c^2 e^{-xt} (e^{-(x+y)t} - 1)}{y^2 (x+y)} + \frac{4a^4 c^3 e^{-xt} (e^{-3yt} - 1)}{3y^2 (x-2y)} \\
& - \frac{4a^4 c^3 e^{-xt} (e^{-(x+y)t} - 1)}{y(x-2y)(x+y)} - \frac{4a^5 c^2 e^{-xt} (e^{-(2x+y)t} - 1)}{y(2x-y)(2x+y)} \\
& + \frac{4a^5 c^2 e^{-xt} (e^{-2yt} - 1)}{2y^2 (2x-y)} - \frac{4a^3 c^4 e^{-xt} (e^{-3yt} - 1)}{3y^3} + \frac{4a^3 c^4 e^{-xt} (e^{-2yt} - 1)}{y^3} \\
& - \frac{4a^5 c^2 e^{-xt} (e^{-xt} - 1)}{xy^2} - \frac{2a^4 c^3 e^{-xt} (e^{-2yt} - 1)}{y^2 (x-2y)} + \frac{4a^4 c^3 e^{-xt} (e^{-xt} - 1)}{xy(x-2y)} \\
& + \frac{2a^5 c^2 e^{-xt} (e^{-2xt} - 1)}{xy(2x-y)} - \frac{4a^5 c^2 e^{-xt} (e^{-yt} - 1)}{y^2 (2x-y)} - \frac{4a^3 c^4 e^{-xt} (e^{-yt} - 1)}{y^3} \\
& + \frac{a^3 c^4 e^{-xt} (e^{(x-4y)t} - 1)}{(x-4y)(x-2y)^2} + \frac{a^3 c^4 e^{-xt} (e^{-2yt} - 1)}{y(x-2y)^2} \\
& + \frac{2a^4 c^3 e^{-xt} (e^{-(x+2y)t} - 1)}{(x-2y)(x+2y)(2x-y)} + \frac{2a^4 c^3 e^{-xt} (e^{(x-3y)t} - 1)}{(2x-y)(x-2y)(x-3y)} \\
& + \frac{4a^3 c^4 e^{-xt} (e^{-3yt} - 1)}{3xy(x-2y)} + \frac{4a^3 c^4 e^{-xt} (e^{(x-3y)t} - 1)}{x(x-2y)(x-3y)} \\
& - \frac{2a^2 c^5 e^{-xt} (e^{(x-4y)t} - 1)}{y(x-4y)(x-2y)} + \frac{2a^2 c^5 e^{-xt} (e^{(x-3y)t} - 1)}{y(x-3y)(x-2y)} \\
& - \frac{a^3 c^4 e^{-xt} (e^{-xt} - 1)}{x(x-2y)^2} - \frac{a^4 c^3 e^{-xt} (e^{-2xt} - 1)}{x(2x-y)(x-2y)} + \frac{2a^4 c^3 e^{-xt} (e^{-yt} - 1)}{y(2x-y)(x-2y)}
\end{aligned} \right]
\end{aligned}$$

Cont.

f₃ Continued

$$\begin{aligned}
& + k^3 b^4 \left[- \frac{4a^3 c^4 e^{-xt} (e^{-(x+y)t} - 1)}{x(x-2y)(x+y)} + \frac{4a^3 c^4 e^{-xt} (e^{-yt} - 1)}{xy(x-2y)} - \frac{a^2 c^5 e^{-xt} (e^{-2yt} - 1)}{y^2(x-2y)} \right. \\
& + \frac{2a^2 c^5 e^{-xt} (e^{-yt} - 1)}{y^2(x-2y)} - \frac{a^5 c^2 e^{-xt} (e^{-3xt} - 1)}{3x(2x-y)^2} + \frac{2a^5 c^2 e^{-xt} (e^{-(x+y)t} - 1)}{(2x-y)^2(x+y)} \\
& - \frac{4a^4 c^3 e^{-xt} (e^{-(2x+y)t} - 1)}{x(2x-y)(2x+y)} + \frac{4a^4 c^3 e^{-xt} (e^{-(x+y)t} - 1)}{x(2x-y)(x+y)} \\
& - \frac{2a^3 c^4 e^{-xt} (e^{-(x+2y)t} - 1)}{y(2x-y)(x+2y)} + \frac{2a^3 c^4 e^{-xt} (e^{-(x+y)t} - 1)}{y(2x-y)(x+y)} \\
& + \frac{a^5 c^2 e^{-xt} (e^{(x-2y)t} - 1)}{(2x-y)^2(x-2y)} + \frac{2a^4 c^3 e^{-xt} (e^{-2yt} - 1)}{xy(2x-y)} \\
& + \frac{4a^4 c^3 e^{-xt} (e^{(x-2y)t} - 1)}{x(2x-y)(x-2y)} - \frac{2a^3 c^4 e^{-xt} (e^{(x-3y)t} - 1)}{y(2x-y)(x-3y)} \\
& + \frac{2a^3 c^4 e^{-xt} (e^{(x-2y)t} - 1)}{y(2x-y)(x-2y)} - \frac{4a^3 c^4 e^{-xt} (e^{-(x+2y)t} - 1)}{x^2(x+2y)} \\
& + \frac{4a^3 c^4 e^{-xt} (e^{-2yt} - 1)}{x^2 y} - \frac{4a^2 c^5 e^{-xt} (e^{-3yt} - 1)}{3xy^2} + \frac{2a^2 c^5 e^{-xt} (e^{-2yt} - 1)}{xy^2} \\
& + \frac{4a^3 c^4 e^{-xt} (e^{(x-2y)t} - 1)}{x^2(x-2y)} - \frac{4a^2 c^5 e^{-xt} (e^{(x-3y)t} - 1)}{xy(x-3y)} \\
& + \frac{4a^2 c^5 e^{-xt} (e^{(x-2y)t} - 1)}{xy(x-2y)} + \frac{ac^6 e^{-xt} (e^{(x-4y)t} - 1)}{y^2(x-4y)} \\
& - \frac{2ac^6 e^{-xt} (e^{(x-3y)t} - 1)}{y^2(x-3y)} + \frac{ac^6 e^{-xt} (e^{(x-2y)t} - 1)}{y^2(x-2y)} \\
& - \frac{a^6 c e^{-yt} (e^{(y-4x)t} - 1)}{x^2(4x-y)} + \frac{2a^6 c e^{-yt} (e^{(y-3x)t} - 1)}{x^2(3x-y)} \\
& - \frac{4a^5 c^2 e^{-yt} (e^{-3xt} - 1)}{3x^2 y} + \frac{4a^5 c^2 e^{-yt} (e^{(y-3x)t} - 1)}{xy(3x-y)} \\
& + \frac{2a^4 c^3 e^{-yt} (e^{-(2x+y)t} - 1)}{x(2x+y)(x-2y)} - \frac{2a^4 c^3 e^{-yt} (e^{(y-3x)t} - 1)}{x(3x-y)(x-2y)} \\
& - \frac{2a^5 c^2 e^{-yt} (e^{(y-4x)t} - 1)}{x(4x-y)(2x-y)} + \frac{a^5 c^2 e^{-yt} (e^{-2xt} - 1)}{x^2(2x-y)} - \frac{4a^4 c^3 e^{-yt} (e^{-3xt} - 1)}{3x^3}
\end{aligned}$$

Cont.

f₃ Continued

$$\begin{aligned}
& + k^3 b^4 + \frac{4a^4 c^3 e^{-yt} (e^{-2xt} - 1)}{x^3} - \frac{10a^3 c^4 e^{-yt} (e^{-(2x+y)t} - 1)}{xy(2x-y)} + \frac{5a^3 c^4 e^{-yt} (e^{-2xt} - 1)}{x^2 y} \\
& - \frac{a^6 c e^{-yt} (e^{(y-2x)t} - 1)}{x^2 (2x-y)} + \frac{2a^5 c^2 e^{-yt} (e^{-2xt} - 1)}{x^2 y} \\
& - \frac{4a^5 c^2 e^{-yt} (e^{(y-2x)t} - 1)}{xy(2x-y)} - \frac{2a^4 c^3 e^{-yt} (e^{-(x+y)t} - 1)}{x(x-2y)(x+y)} \\
& + \frac{2a^4 c^3 e^{-yt} (e^{(y-2x)t} - 1)}{x(2x-y)(x-2y)} + \frac{2a^5 c^2 e^{-yt} (e^{(y-3x)t} - 1)}{x(3x-y)(2x-y)} \\
& - \frac{2a^5 c^2 e^{-yt} (e^{-xt} - 1)}{x^2 (2x-y)} - \frac{4a^4 c^3 e^{-yt} (e^{-xt} - 1)}{x^3} + \frac{10a^3 c^4 e^{-yt} (e^{-(x+y)t} - 1)}{xy(x+y)} \\
& - \frac{10a^3 c^4 e^{-yt} (e^{-xt} - 1)}{x^2 y} - \frac{4a^4 c^3 e^{-yt} (e^{-(2x+y)t} - 1)}{y^2 (2x+y)} + \frac{4a^4 c^3 e^{-yt} (e^{-2xt} - 1)}{xy^2} \\
& + \frac{4a^3 c^4 e^{-yt} (e^{-(x+2y)t} - 1)}{y(x-2y)^2} - \frac{2a^3 c^4 e^{-yt} (e^{-2xt} - 1)}{xy(x-2y)} - \frac{4a^4 c^3 e^{-yt} (e^{-3xt} - 1)}{3xy(2x-y)} \\
& + \frac{4a^4 c^3 e^{-yt} (e^{-(x+y)t} - 1)}{y(2x-y)(x+y)} - \frac{4a^2 c^5 e^{-yt} (e^{-(x+2y)t} - 1)}{y^2 (x+2y)} \\
& + \frac{8a^2 c^5 e^{-yt} (e^{-(x+y)t} - 1)}{y^2 (x+y)} - \frac{4a^4 c^3 e^{-yt} (e^{(y-2x)t} - 1)}{y^2 (2x-y)} \\
& - \frac{4a^3 c^4 e^{-yt} (e^{-(x+y)t} - 1)}{y(x-2y)(x+y)} + \frac{4a^3 c^4 e^{-yt} (e^{(y-2x)t} - 1)}{y(2x-y)(x-2y)} \\
& + \frac{4a^4 c^3 e^{-yt} (e^{(y-3x)t} - 1)}{y(3x-y)(2x-y)} - \frac{4a^4 c^3 e^{-yt} (e^{-xt} - 1)}{xy(2x-y)} - \frac{4a^2 c^5 e^{-yt} (e^{-xt} - 1)}{xy^2} \\
& - \frac{a^2 c^5 e^{-yt} (e^{-3yt} - 1)}{3y(x-2y)^2} + \frac{2a^2 c^5 e^{-yt} (e^{-(x+y)t} - 1)}{(x-2y)^2 (x+y)} \\
& + \frac{2a^3 c^4 e^{-yt} (e^{-(2x+y)t} - 1)}{(2x+y)(2x-y)(x-2y)} - \frac{a^3 c^4 e^{-yt} (e^{-2yt} - 1)}{y(2x-y)(x-2y)} \\
& + \frac{4a^2 c^5 e^{-yt} (e^{-(x+2y)t} - 1)}{x(x-2y)(x+2y)} - \frac{2a^2 c^5 e^{-yt} (e^{-2yt} - 1)}{xy(x-2y)} + \frac{2ac^6 e^{-yt} (e^{-3yt} - 1)}{3y^2 (x-2y)} \\
& - \frac{ac^6 e^{-yt} (e^{-2yt} - 1)}{y^2 (x-2y)} - \frac{a^2 c^5 e^{-yt} (e^{(y-2x)t} - 1)}{(2x-y)(x-2y)^2}
\end{aligned}$$

Cont.

f_3 Continued

$$\begin{aligned}
 + k^3 b^4 & \left[\begin{aligned}
 & - \frac{2a^3 c^4 e^{-yt} (e^{(y-3x)t} - 1)}{(3x-y)(2x-y)(x-2y)} + \frac{2a^3 c^4 e^{-yt} (e^{-xt} - 1)}{x(2x-y)(x-2y)} - \frac{2a^2 c^5 e^{-yt} (e^{-2xt} - 1)}{x^2(x-2y)} \\
 & + \frac{4a^2 c^5 e^{-yt} (e^{-xt} - 1)}{x^2(x-2y)} - \frac{2ac^6 e^{-yt} (e^{-(x+y)t} - 1)}{y(x-2y)(x+y)} + \frac{2ac^6 e^{-yt} (e^{-xt} - 1)}{xy(x-2y)} \\
 & - \frac{a^4 c^3 e^{-yt} (e^{(y-4x)t} - 1)}{(4x-y)(2x-y)^2} + \frac{a^4 c^3 e^{-yt} (e^{-2xt} - 1)}{x(2x-y)^2} - \frac{4a^3 c^4 e^{-yt} (e^{-3xt} - 1)}{3x^2(2x-y)} \\
 & + \frac{2a^3 c^4 e^{-yt} (e^{-2xt} - 1)}{x^2(2x-y)} - \frac{2a^2 c^5 e^{-yt} (e^{-(2x+y)t} - 1)}{y(2x-y)(2x+y)} + \frac{a^2 c^5 e^{-yt} (e^{-2xt} - 1)}{xy(x-2y)} \\
 & - \frac{a^4 c^3 e^{-yt} (e^{-yt} - 1)}{y(2x-y)^2} + \frac{4a^3 c^4 e^{-yt} (e^{-(x+y)t} - 1)}{x(2x-y)(x+y)} - \frac{4a^3 c^4 e^{-yt} (e^{-yt} - 1)}{xy(2x-y)} \\
 & + \frac{a^2 c^5 e^{-yt} (e^{-2yt} - 1)}{y^2(2x-y)} - \frac{2a^2 c^5 e^{-yt} (e^{-yt} - 1)}{y^2(2x-y)} - \frac{4a^2 c^5 e^{-yt} (e^{-(2x+y)t} - 1)}{x^2(2x+y)} \\
 & + \frac{8a^2 c^5 e^{-yt} (e^{-(x+y)t} - 1)}{x^2(x+y)} - \frac{4ac^6 e^{-yt} (e^{-(x+2y)t} - 1)}{xy(x+2y)} \\
 & + \frac{4ac^6 e^{-yt} (e^{-(x+y)t} - 1)}{xy(x+y)} - \frac{4a^2 c^5 e^{-yt} (e^{-yt} - 1)}{x^2 y} + \frac{4ac^6 e^{-yt} (e^{-2yt} - 1)}{2xy^2} \\
 & - \frac{4ac^6 e^{-yt} (e^{-yt} - 1)}{xy^2} - \frac{c^7 e^{-yt} (e^{-3yt} - 1)}{3y^3} + \frac{c^7 e^{-yt} (e^{-2yt} - 1)}{y^3} \\
 & - \frac{c^7 e^{-yt} (e^{-yt} - 1)}{y^3}
 \end{aligned} \right]
 \end{aligned}$$

f_4 is shown in the Appendix.

For given numerical values of x and y one could probably show uniform convergence for this result. It is much easier to depend upon Poincaré's theorem for k small enough to assure convergence.

8. ASSIMILATION OF SECULAR TERMS

Transform solutions are formal solutions of the differential equation. When repeated roots occur in the operational transform secular terms appear in the solution. In cases similar to the pendulum where the period (for oscillatory functions) is known the summation of p^{-n} may be carried out by the computer, or if a problem is first solved by hand the summation can be carried out by the machine but there would be no point in wasting machine time.

One may adopt the point of view that with machine solutions a sufficient number of secular terms may easily be obtained to approximate the solution or one may attempt to recognize the secular terms in such a way as to assimilate them into the solution as known functions. As an example of the two methods consider the equation

$$f'' + \omega^2 f = -kf^2. \quad (69)$$

A formal solution to this equation may be obtained by the derivative process or by the expansion method.

The derivative process, after summation and sorting gives Equation (70) directly while the expansion method (after appropriate changes in constants) gives Equation (71) directly.

$$\begin{aligned} F(p) = & \frac{ap^2}{p^2 + \omega^2} - \frac{ka^2(p^2 + 2\omega^2)}{(p^2 + \omega^2)(p^2 + 4\omega^2)} \\ & + \frac{k^2 a^3 (2p^4 + 10\omega^2 p^2 - 12\omega^4)}{(p^2 + \omega^2)^2 (p^2 + 4\omega^2) (p^2 + 9\omega^2)} \\ & - \frac{k^3 a^4 (10p^6 + 150\omega^2 p^4 + 440\omega^4 p^2 + 1200\omega^6)}{(p^2 + \omega^2)^2 (p^2 + 4\omega^2)^2 (p^2 + 9\omega^2) (p^2 + 16\omega^2)} \\ & + k^4 a^5 (80p^{12} + 3,880\omega^2 p^{10} + 65,040\omega^4 p^8 + 444,440\omega^6 p^6 \\ & \quad + 942,880\omega^8 p^4 - 3,040,320\omega^{10} p^2 - 5,760,000\omega^{12}) \\ & \quad \frac{}{(p^2 + \omega^2)^3 (p^2 + 4\omega^2)^2 (p^2 + 9\omega^2)^2 (p^2 + 16\omega^2)^2 (p^2 + 25\omega^2)} \\ & + + + \end{aligned} \quad (70)$$

When $F(p)$ is evaluated as a time function:

$$\begin{aligned}
 f(t) = & a \cos \omega t \\
 & + \frac{ka^2}{6\omega^2} (\cos 2\omega t + 2 \cos \omega t - 3) \\
 & + \frac{k^2 a^3}{144\omega^4} (3 \cos 3\omega t + 16 \cos 2\omega t + 29 \cos \omega t + 60 \omega t (\sin \omega t) - 48) \\
 & + \frac{k^3 a^4}{2160\omega^6} (5 \cos 4\omega t + 45 \cos 3\omega t + 480 \cos 2\omega t + 595 \cos \omega t \\
 & \quad + 300\omega t (\sin 2\omega t) + 900 \omega t (\sin \omega t) - 1125)
 \end{aligned} \tag{71}$$

This equation may be rewritten as

$$\begin{aligned}
 f(t) = & \left(a + \frac{ka^2}{3\omega^2} + \frac{29k^2 a^3}{144\omega^4} + \frac{595k^3 a^4}{2160\omega^6} + + \right) \cos \omega t \\
 & + \left(\frac{ka^2}{6\omega^2} + \frac{16k^2 a^3}{144\omega^4} + \frac{480k^3 a^4}{2160\omega^6} + + \right) \cos 2\omega t \\
 & + \left(\frac{3k^2 a^3}{144\omega^4} + \frac{45k^3 a^4}{2160\omega^6} + + + \right) \cos 3\omega t \\
 & + \left(\frac{5k^3 a^4}{2160\omega^6} + + + \right) \cos 4\omega t \\
 & + \left[(60\omega t) \frac{k^2 a^3}{144\omega^4} + (900\omega t) \frac{k^3 a^4}{2160\omega^6} \right] \sin \omega t \\
 & + \frac{(300\omega t) k^3 a^4}{2160\omega^6} \sin 2\omega t \\
 & - \left[\frac{ka^2}{2\omega^2} + \frac{48k^2 a^3}{144\omega^4} + \frac{1125k^3 a^4}{2160\omega^6} + + + \right].
 \end{aligned} \tag{72}$$

To assimilate the secular terms ($\sin \omega t$ and $\sin 2\omega t$) the relation

$$\begin{aligned}
 \cos(\omega + \beta) t &= \cos \omega t \cos \beta t - \sin \omega t \sin \beta t \\
 &= \cos \omega t \left(1 - \frac{\beta^2 t^2}{2} + \frac{\beta^4 t^4}{4} + \dots \right) \\
 &= \sin \omega t \left(\frac{\beta t}{1} - \frac{\beta^2 t^3}{3} + \dots \right)
 \end{aligned} \tag{73}$$

is compared with the $\cos \omega t$ and $\sin \omega t$ terms of Equation (72).

$$\begin{aligned}
 &\left(a + \frac{k a^2}{3\omega^2} + \frac{29k^2 a^3}{144\omega^4} + \dots \right) \cos \omega t \\
 &+ t \left[(60\omega) \frac{k^2 a^3}{144\omega^4} + \frac{(900\omega) k^3 a^4}{2160\omega^6} \right] \sin \omega t .
 \end{aligned} \tag{74}$$

Divide the coefficient of ($t \sin \omega t$) by the coefficient of $\cos \omega t$. The result is:

$$\frac{5 k^2 a^2}{12\omega^3} + \frac{5 k^3 a^3}{18\omega^5} \tag{75}$$

therefore in Equation (73) β is the negative of this value.

The new rotational velocity is

$$\omega + \beta = \omega \left(1 - \frac{5k^2 a^2}{12\omega^4} - \frac{5k^3 a^3}{18\omega^6} + \dots \right) = \omega_1 . \tag{76}$$

The coefficient of ($t \sin 2\omega t$) when divided by the leading coefficient of $\cos 2\omega t$ gives

$$2 \left(\frac{5k^2 a^2 \omega}{12\omega^4} \right)$$

which matches the second term on the right of (76) since this is double frequency.

The period is

$$T = \frac{2\pi}{\omega_1} \quad (77)$$

Figure 8 shows the graphical result of the solution by numerical integration for $k = 0.1$, $\omega = 1$ and $f(0) = 1$.

$$T \cong \frac{2\pi}{(1) \left(1 - \frac{5(0.01)}{12} - \frac{5(0.001)}{18}\right)} = \frac{6.283}{1 - 0.0045} \cong 6.31 \text{ seconds}$$

which appears to check the period in the figure.

In Figure 9 $f(0)$ has been changed to 3. In this case

$$T \cong \frac{2\pi}{1 - \frac{5(0.01)(3)^2}{12} - \frac{5(0.001)(3)^3}{18}} = 6.58 \text{ seconds}$$

which is probably very close to the period of the numerical solution for this case.

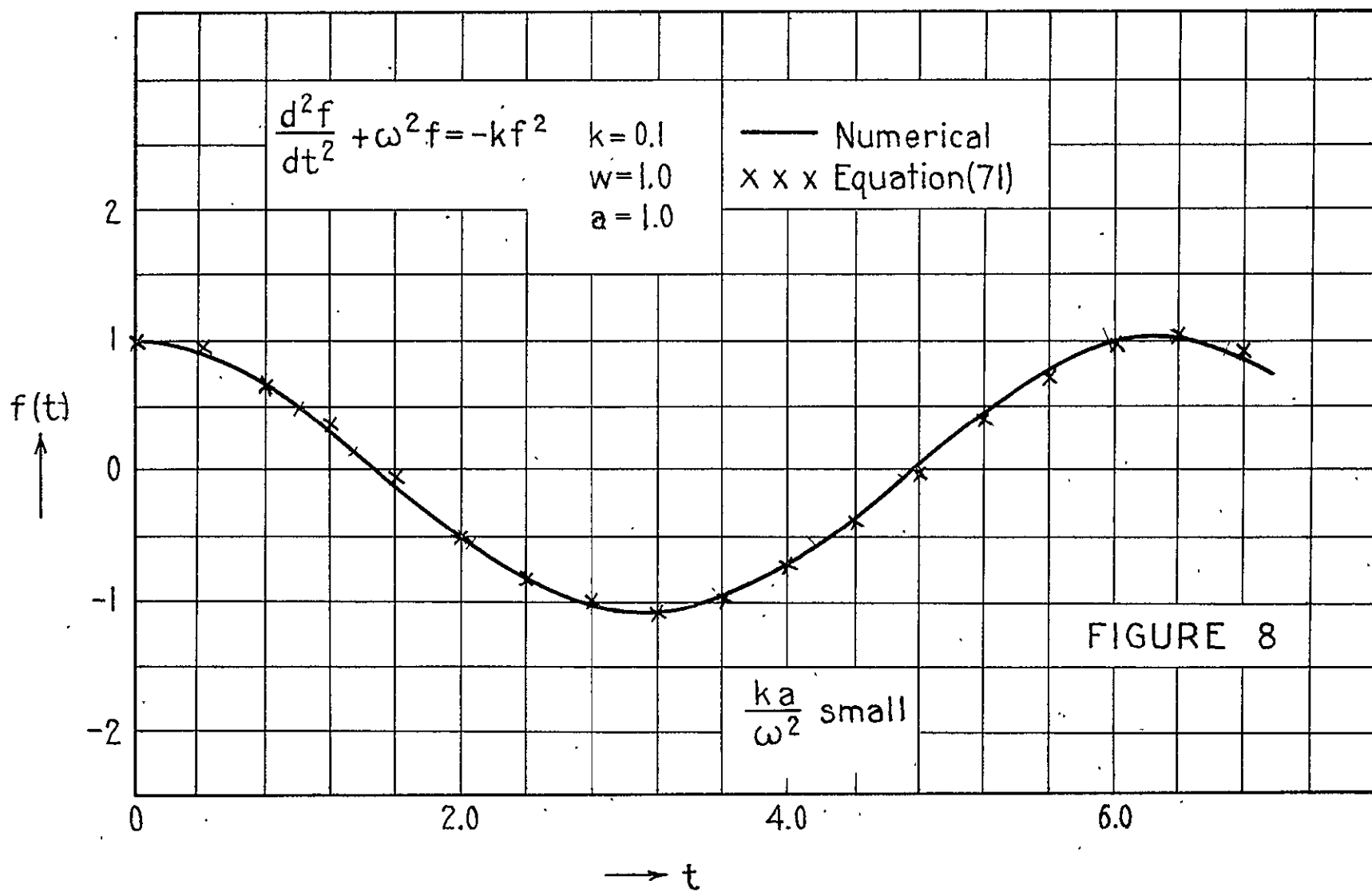
The solid line therefore represents the solution with the secular terms assimilated for both figures.

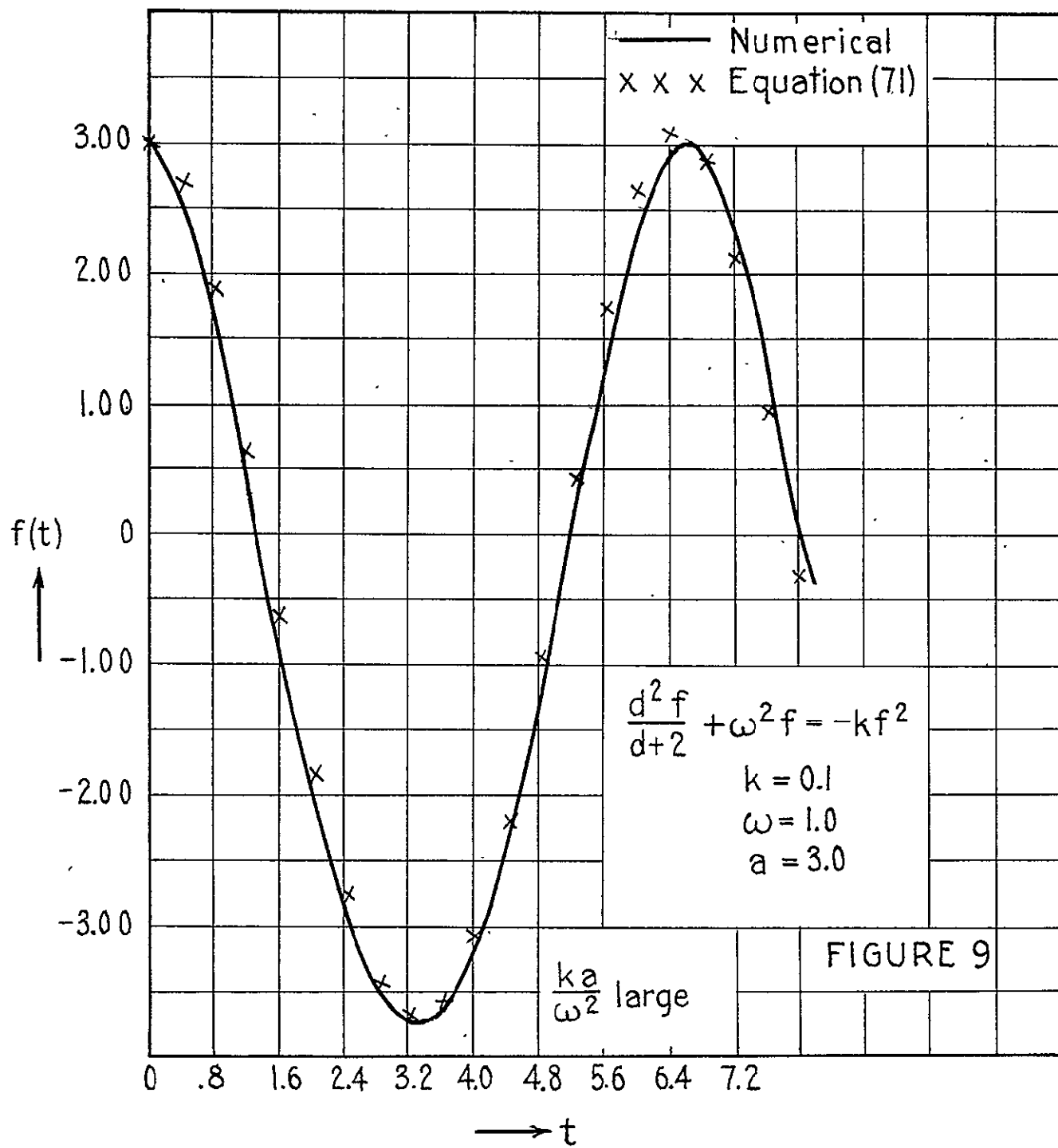
Equation (71) may therefore be rewritten dropping the sine terms and changing ω to ω_1 in the cosine terms. The ω terms in the denominators are unchanged.

$$\begin{aligned} f(t) = & \frac{ka^2}{6\omega} (\cos 2\omega_1 t + 2 \cos \omega_1 t - 3) \\ & + \frac{k^2 a^3}{144\omega^4} (3 \cos 3\omega_1 t + 16 \cos 2\omega_1 t + 29 \cos \omega_1 t - 48) \\ & + \frac{k^3 a^4}{2160\omega^6} (5 \cos 4\omega_1 t + 45 \cos 3\omega_1 t + 480 \cos 2\omega_1 t \\ & \quad + 595 \cos \omega_1 t - 1125) \end{aligned} \quad (78)$$

The crosses in both Figure 8 and Figure 9 indicate the equivalent solutions for Equation (71) including the secular terms as shown in this equation.

Convergence is assured if the factor ka/ω^2 is small enough.





9. THE VAN DER POL EQUATION

The van der Pol equation is perhaps the most interesting equation studied.

$$f'' - A(1 - f^2) f' + Bf = 0 \quad (79)$$

The work on the computer was actually carried out with two different equations

$$f'' - Af' + Bf = -kf^2 f' \quad (80)$$

where $k = A$ in the solution

and

$$f'' - 2Xf' + (X^2 - Y^2) f = 0 \quad (81)$$

where

$$X = \frac{A - f^2 k}{2} \quad Y = \sqrt{\frac{(A - f^2 k)^2}{2} - B} \quad (82)$$

X and Y are time functions.

In the final results

$$X(0) = \frac{A - a^2 k}{2} = x \quad \text{and}$$

$$Y(0) = \sqrt{\frac{(A - a^2 k)^2}{2} - B} = y$$

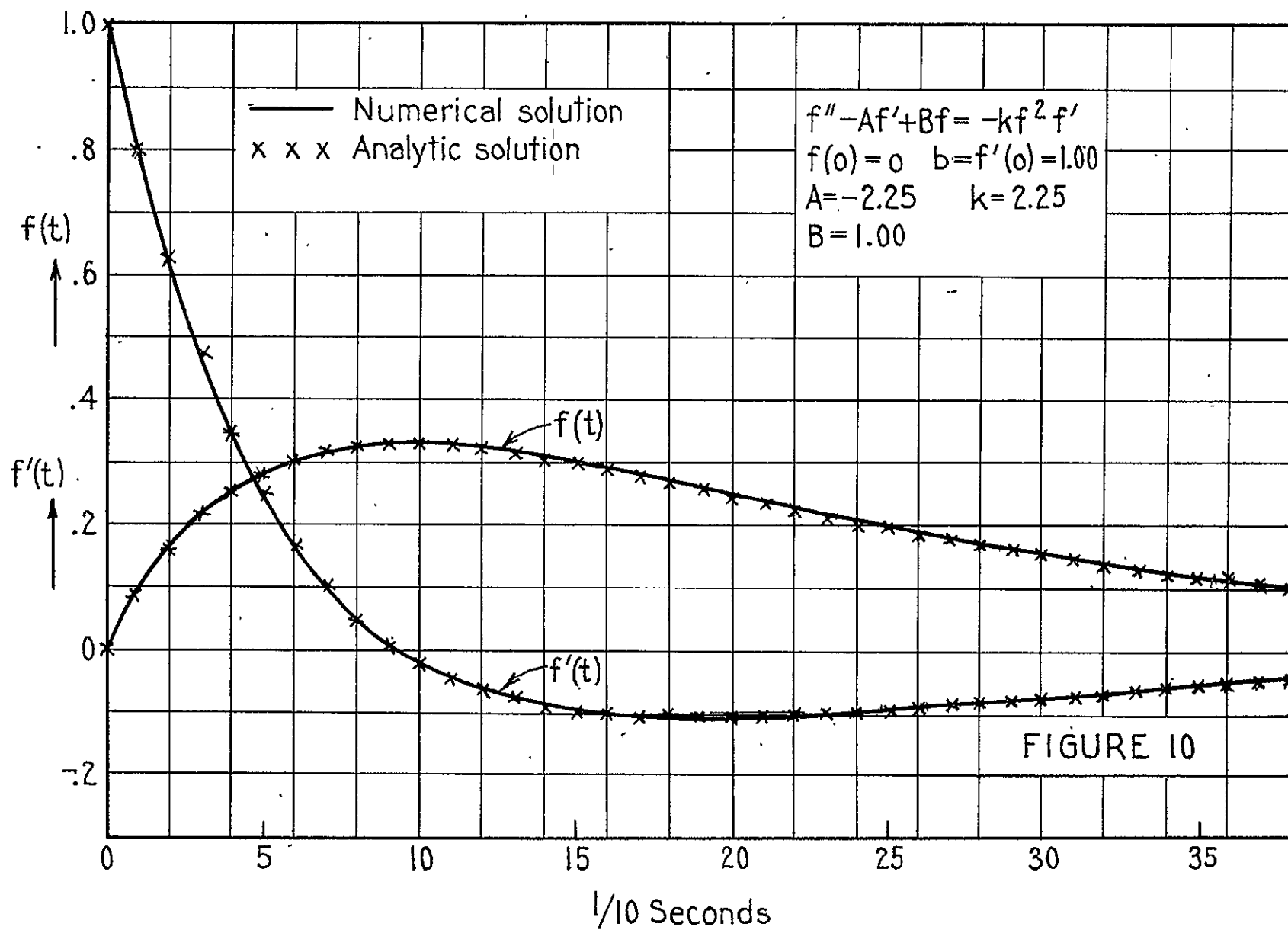
where $a = f(0)$ and $b = f'(0)$. Again $k = A$ for the van der Pol Equation.

The first four operational terms for Equation (80) for the case $f(0) = 0$ and $f'(0) = b$ are listed as Equation (83).

$$\begin{aligned}
 F(p) = & \frac{bp}{(p^2 - Ap + B)} - \frac{2Kb^3p^2}{(p^2 - Ap + B)(p^2 - 3Ap + B + 2A^2)(p^2 - 3Ap + B)} \\
 & + \frac{60K^2b^5p^5 + 348K^2b^5Bp^3 - 252K^2b^5Ap^4 - 480K^2b^5ABp^2 + 240K^2b^5A^2p^3}{(p^2 - Ap + B)(p^2 - 3Ap + B + 2A^2)(p^2 - 3Ap + 9B)(p^2 - 5Ap + B + 6A^2)} \\
 & \frac{(p^2 - 5Ap + 25B)(p^2 - 5Ap + 9B + 4A^2)}{(p^2 - 5Ap + 25B)(p^2 - 5Ap + 9B + 4A^2)(p^2 - 7Ap + B + 12A^2)} \\
 & - \frac{[6840K^3b^7p^{10} - 112,680K^3b^7Ap^9 + 743,352K^3b^7A^2p^8 + 209,160K^3b^7Bp^8 \\
 & - 2,512,536K^3b^7A^3p^7 - 2,493,648K^3b^7ABp^7 + 4,587,024K^3b^7A^4p^6 \\
 & + 11,329,032K^3b^7A^2Bp^6 + 1,646,280K^3b^7B^2p^6 - 4,290,240K^3b^7A^5p^5 \\
 & - 24,411,264K^3b^7A^3Bp^5 - 13,073,832K^3b^7AB^2p^5 + 1,612,800K^3b^7A^6p^4 \\
 & + 24,876,480K^3b^7A^4Bp^4 + 35,517,072K^3b^7A^2B^2p^4 + 4,005,720K^3b^7B^3p^4 \\
 & - 9,676,800K^3b^7A^5Bp^3 - 38,425,920K^3b^7A^3B^2p^3 - 14,964,480K^3b^7AB^3p^3 \\
 & + 14,112,000K^3b^7A^4B^2p^2 + 12,868,800K^3b^7A^2B^3p^2 - 756,000K^3b^7B^4p^2]}{(p^2 - Ap + B)(p^2 - 3Ap + B + 2A^2)(p^2 - 3Ap + 9B)(p^2 - 5Ap + B + 6A^2)} \\
 & \frac{(p^2 - 5Ap + 25B)(p^2 - 5Ap + 9B + 4A^2)(p^2 - 7Ap + B + 12A^2)}{(p^2 - 7Ap + 25B + 6A^2)(p^2 - 7Ap + 49B)(p^2 - 7Ap + 9B + 10A^2)} \quad (83)
 \end{aligned}$$

This was the first second order equation attempted and no summation was obtained for the general case where both initial conditions were present or for the case $f(0) = a$ and $f'(0) = 0$.

Figure 10 shows the analytic solution for the case $f(0) = 0$, $f'(0) = 1$, $K = 2.25$, $B = 1$ and $A = -2.25$. (Since A is negative the equation is not in the standard form for the van der Pol equation.) The solid lines, as usual, are the numerical solutions. The agreement is surprisingly good since both numerical and analytic results are approximations.



The first three terms of the operational solution for Equation (81) for $f(0) = a$ and $f'(0) = 0$ are

$$\begin{aligned}
 F(a,p) &= \frac{a(p^2 - 2xp)}{M_{11}} \\
 &+ Ka^3 \left[\frac{-6x^4p^3 + 12x^2y^2p^3 - 6y^4p^3 + 44x^5p^2 - 88x^3y^2p^2}{(M_{11})^2 M_{31} M_{33}} \right. \\
 &\quad \left. -252x^6p^{12} + 756x^4y^2p^{12} - 756x^2y^4p^{12} + 252y^6p^{12} + 10,360x^7p^{11} \right. \\
 &\quad -31,080x^5y^2p^{11} + 31,080x^3y^4p^{11} - 10,360xy^6p^{11} - 183,148x^8p^{10} \\
 &\quad + 553,008x^6y^2p^{10} - 560,136x^4y^4p^{10} + 193,840x^2y^6p^{10} - 3564y^8p^{10} \\
 &\quad + 1,835,040x^9p^9 - 5,623,488x^7y^2p^9 + 5,860,224x^5y^4p^9 - 2,190,144x^3y^6p^9 \\
 &\quad + 118,368xy^8p^9 - 11,588,472x^{10}p^8 + 36,430,840x^8y^2p^8 - 39,775,728x^6y^4p^8 \\
 &\quad + 16,626,864x^4y^6p^8 - 1,707,544x^2y^8p^8 + 48,416,464x^{11}p^7 - 158,060,048x^9y^2p^7 \\
 &\quad + 184,083,360x^7y^4p^7 - 88,054,432x^5y^6p^7 + 14,016,656x^3y^8p^7 - 402,000xy^{10}p^7 \\
 &\quad + K^2a^5 \left[-136,102,200x^{12}p^6 + 467,291,696x^{10}y^2p^6 - 589,682,024x^8y^4p^6 \right. \\
 &\quad + 326,341,152x^6y^6p^6 - 72,315,272x^4y^8p^6 + 4,489,904x^2y^{10}p^6 + 255,160,352x^{13}p^5 \\
 &\quad - 932,617,728x^{11}y^2p^5 + 1,290,557,984x^9y^4p^5 - 828,061,440x^7y^6p^5 \\
 &\quad + 239,608,416x^5y^8p^5 - 25,137,920x^3y^{10}p^5 + 490,336xy^{12}p^5 - 306,409,612x^{14}p^4 \\
 &\quad + 1,204,893,748x^{12}y^2p^4 - 1,842,131,132x^{10}y^4p^4 + 1,364,410,308x^8y^6p^4 \\
 &\quad - 493,254,820x^6y^8p^4 + 75,809,564x^4y^{10}p^4 - 3,335,444x^2y^{12}p^4 + 213,491,832x^{15}p^3 \\
 &\quad - 910,349,256x^{13}y^2p^3 + 1,541,774,040x^{11}y^4p^3 - 1,307,094,504x^9y^6p^3 \\
 &\quad + 571,950,696x^7y^8p^3 - 119,115,096x^5y^{10}p^3 + 9,538,632x^3y^{12}p^3 - 65,678,940x^{16}p^2 \\
 &\quad + 304,917,120x^{14}y^2p^2 - 570,702,960x^{12}y^4p^2 + 547,560,000x^{10}y^6p^2 \\
 &\quad - 282,794,760x^8y^8p^2 + 75,559,680x^6y^{10}p^2 - 9,215,280x^4y^{12}p^2 + 14,040y^{10}p^8 \\
 &\quad \left. - 23,256y^{12}p^6 + 17,388y^{14}p^4 - 196,344xy^{14}p^3 + 360,000x^2y^{14}p^2 - 4860y^{16}p^2 \right] \\
 &\quad (M_{11})^3 (M_{31})^2 (M_{33})^2 M_{51} M_{53} M_{55}
 \end{aligned}
 \tag{84}$$

where

$$M_{wn} \equiv (p^2 - 2wpx + w^2x^2 - n^2y^2)$$

and the first four terms for $f(0) = a$ and $f'(0) = b$ are

$$\begin{aligned}
 F(b,p) = & \frac{bp}{M_{11}} - \frac{2Kb^3p^2}{M_{11}M_{31}M_{33}} \\
 & + \frac{K^2b^5(60p^5 - 504xp^4 + 1308x^2p^3 - 348y^2p^3 - 960x^3p^2 + 960xy^2p^2)}{M_{11}M_{31}M_{33}M_{51}M_{53}M_{55}} \\
 & + K^3b^7 \frac{
 \begin{aligned}
 & - 6840p^{10} + 225,360xp^9 - 3,182,568x^2p^8 + 209,160y^2p^8 + 25,087,584x^3p^7 \\
 & - 4,987,296xy^2p^7 - 120,354,792x^4p^6 + 48,608,688x^2y^2p^6 - 1,646,280y^4p^6 \\
 & + 358,725,456x^5p^5 - 247,585,440x^3y^2p^5 + 26,147,664xy^4p^5 \\
 & - 647,316,888x^6p^4 + 694,177,416x^4y^2p^4 - 154,085,448x^2y^4p^4 \\
 & + 4,005,720y^6p^4 + 646,993,920x^7p^3 - 1,014,259,200x^5y^2p^3 \\
 & + 397,194,240x^3y^4p^3 - 29,928,960xy^6p^3 - 276,511,200x^8p^2 \\
 & + 602,985,600x^6y^2p^2 - 375,681,600x^4y^4p^2 + 48,451,200x^2y^6p^2 \\
 & + 756,000y^8p^2
 \end{aligned}
 }{M_{11}M_{31}M_{33}M_{51}M_{53}M_{55}M_{71}M_{73}M_{75}M_{77}}
 \end{aligned}$$

Here

$$M_{wn} \equiv (p^2 - 2wxp + w^2x^2 - n^2y^2) \quad (85)$$

No satisfactory evaluation of Equation (84) has been obtained due to the large number of multiple roots. A computer program is available to evaluate this equation as a time function for given numerical values of the constants and the initial condition but unfortunately there appear to be too few secular terms to approximate the function.

The expansion type of solution might be very helpful here in identifying the series of secular terms and attempting to assimilate them.

Equations (84) and (85) were generated with the idea of simplifying the root structure of Equation (83). Notice that the roots for Equation (84) are

$$\begin{aligned}
 m_{\omega n} &= (p^2 - 2\omega xp + \omega^2 x^2 - n^2 y^2) \\
 &= \left[p^2 - \omega (A - a^2 k)p \right. \\
 &\quad \left. + (\omega^2 - n^2) \left(\frac{A - a^2 k}{2} \right)^2 + B \right]
 \end{aligned} \tag{86}$$

where ω and n take on the values shown in (84). Since $a = f(0)$ this means that at least one initial value occurs in the root structure.

Equation (85) is the same as Equation (83) if the values for x and y are substituted in the equation $f(0) = a = 0$.

Poincaré's theorem insures convergence for small values of k so the result is an analytic function.

10. DUFFING'S EQUATION

The general form of Duffing's Equation is usually given as:

$$f'' + cf' + rf = -kf^3 + G \cos \omega t. \quad (87)$$

If the general solution for this equation was of sufficient importance it could no doubt be obtained. In Heaviside notation with two initial conditions the original equation becomes

$$(p+x)(p+y)f = ap^2 + bp + (x+y)ap + G \cos \omega t - kf^3 \quad (88)$$

where $a = f(0)$ and $b = f'(0)$.

$$f = \frac{ap^2 + bp + (x+y)ap}{(p+x)(p+y)} + \frac{G \cos \omega t}{(p+x)(p+y)} - \left[\frac{1}{(p+x)(p+y)} \right] \left[kf^3 \right]. \quad (89)$$

Even an approximate solution through the k^3 term for this equation would take a great deal of time on a large computer. To reduce computer time take $a = b = 0$ and $y = x$ so the equation now becomes:

$$f = \frac{G \cos \omega t}{p^2 + x^2} - \frac{k f^3}{p^2 + x^2}. \quad (90)$$

The first term on the right hand side is

$$f_1 = \frac{G (\cos \omega t - \cos xt)}{(x^2 - \omega^2)} \quad (91)$$

$$f_2 = -k \left[\frac{1}{p^2 + x^2} \right] \left[f_1^3 \right] \quad (92)$$

$$f_3 = -k \left[\frac{1}{p^2 + x^2} \right] \left[3f_1^2 f_2 + 3f_1 f_2^2 + f_2^3 \right] \quad (93)$$

$$f = f_1 + f_2 + f_3 + \dots \quad (94)$$

Computer results give:

$$\begin{aligned}
 f_2 = & + \frac{k G^3}{8x (x^2 - \omega^2)^3} \\
 & \left[\left\{ \frac{9x}{x^2 - \omega^2} + \frac{18x}{9x^2 - \omega^2} + \frac{2x}{x^2 - 9\omega^2} \right. \right. \\
 & \quad \left. \left. + \frac{1}{4x} \right\} \left\{ \cos xt \right\} + 9t \sin xt \right. \\
 & \quad - \frac{1}{4x} \cos 3xt - \frac{18x}{(x^2 - \omega^2)^2} \cos \omega t \\
 & \quad - \frac{2x}{(x^2 - 9\omega^2)} \cos 3\omega t + \frac{3x}{2\omega (x-\omega)} \cos (x-2\omega)t \\
 & \quad - \frac{3x}{2\omega (x+\omega)} \cos (x+2\omega)t \\
 & \quad + \frac{6x}{(3x+\omega)(x+\omega)} \cos (2x+\omega)t \\
 & \quad \left. + \frac{6x}{(3x-\omega)(x-\omega)} \cos (2x-\omega)t \right] \quad (95)
 \end{aligned}$$

f_3 (secular terms only)

$$\begin{aligned}
 & = \frac{-k^2 G^5}{(2)^6 x^2 (x^2 - \omega^2)^5} \left[-\frac{81t^2}{2} \cos xt \right. \\
 & \quad + \left\{ \frac{87t}{2x} + \frac{324xt}{x^2 - \omega^2} + \frac{378xt}{9x^2 - \omega^2} + \frac{30xt}{x^2 - 9\omega^2} \right\} \sin xt \\
 & \quad - \frac{27t}{4x} \sin 3xt - \frac{27xt}{2\omega (x+\omega)} \sin (x+2\omega)t \\
 & \quad + \frac{108xt}{(3x-\omega)(x-\omega)} \sin (2x-\omega)t + \frac{108xt \sin (2x+\omega)t}{(3x+\omega)(x+\omega)} \\
 & \quad \left. + \frac{27xt}{2\omega (x-\omega)} \sin (x-2\omega)t \right]. \quad (96)
 \end{aligned}$$

Rearranging the terms in (91), (95) and (96):

$$\begin{aligned}
 f = & \frac{G \cos \omega t}{(x^2 - \omega^2)} - \frac{9 k G^3}{4 (x^2 - \omega^2)^4} \cos \omega t - \frac{k G^3 \cos 3\omega t}{4 (x^2 - \omega^2)^3 (x^2 - 9\omega^2)} \\
 & - \frac{G}{(x^2 - \omega^2)} \left[\cos xt \left\{ 1 - \left(\frac{9 k G^2 t}{8x (x^2 - \omega^2)^2} \right)^2 \frac{1}{2!} + + \right\} \right. \\
 & \left. - \sin xt \cdot \frac{9 k G^2 t}{8x (x^2 - \omega^2)^3} + + + \right] - \frac{k G^3}{32x^2 (x^2 - \omega^2)^3} \left[\cos 3xt (1 + +) \right. \\
 & \left. - 3 \left(\frac{9 k G^2 t}{8x (x^2 - \omega^2)^2} + + + \right) \sin 3xt \right] \\
 & + \frac{3 k G^3}{16\omega (x - \omega) (x^2 - \omega^2)^3} \left[\cos (x - 2\omega)t (1 + +) \right. \\
 & \left. - \left(\frac{9 k G^2 t}{8x (x^2 - \omega^2)^2} + + + \right) \sin (x - 2\omega)t \right] - \\
 & - \frac{3 k G^3}{16\omega (x + \omega) (x^2 - \omega^2)^3} \left[\cos (x + 2\omega)t (1 + +) \right. \\
 & \left. - \left(\frac{9 k G^2 t}{8x (x^2 - \omega^2)^2} + + + \right) \sin (x + 2\omega)t \right] \\
 & + \frac{9 k G^3}{8 (x^2 - \omega^2)^4} \left[\cos xt (1 + +) - 4 \left(\frac{9 k G^2 t}{8x (x^2 - \omega^2)^2} + + \right) \sin xt \right] \\
 & + \frac{3 k G^3}{4 (3x + \omega) (x + \omega) (x^2 - \omega^2)^3} \left[\cos (2x + \omega)t (1 + +) \right.
 \end{aligned}$$

$$\begin{aligned}
& -2 \left(\frac{9 k G^2 t}{8x (x^2 - \omega^2)^2} + + \right) \sin (2x + \omega)t \Big] \\
& + \frac{3 k G^3}{4 (3x - \omega) (x - \omega) (x^2 - \omega^2)^3} \left[\cos (2x - \omega)t (1 + +) \right. \\
& \left. -2 \left(\frac{9 k G^2 t}{8x (x^2 - \omega^2)^2} + + \right) \sin (2x - \omega)t \right] \\
& + \frac{k G^3}{4 (x^2 - \omega^2)^3 (x^2 - 9\omega^2)} \left[\cos xt (1 + +) - \left(\frac{15 k G^2 t}{8x (x^2 - \omega^2)^2} + + \right) \sin \omega t \right] \\
& + \frac{k G^3}{32x^2 (x^2 - \omega^2)^3} \left[\cos xt (1 + +) - \left(\frac{87 k G^2 t}{4x (x^2 - \omega^2)^2} + + \right) \sin xt \right] \\
& + \frac{9 k G^3}{4 (x^2 - \omega^2)^3 (9x^2 - \omega^2)} \left[\cos xt (1 + +) - \left(\frac{21 k G^2 t}{8x (x^2 - \omega^2)^2} + + \right) \sin xt \right]
\end{aligned} \tag{97}$$

With the substitutions:

$$\frac{9 k G^2}{8x (x^2 - \omega^2)^2} = \beta \tag{98}$$

$$\frac{G}{x^2 - \omega^2} = z \tag{99}$$

$$(x^2 - \omega^2) = m \tag{100}$$

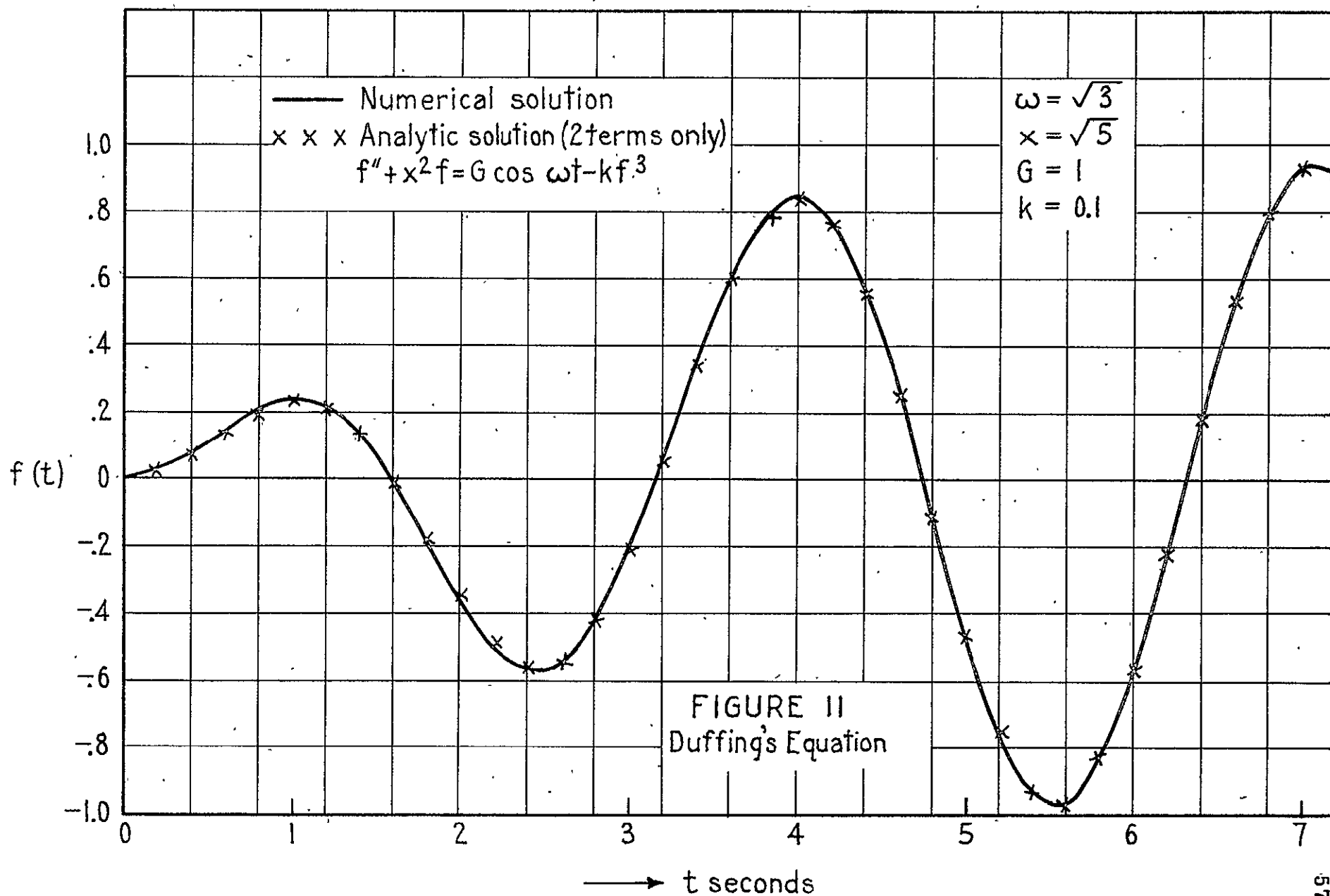
a very rough approximation for the first two terms gives

$$\begin{aligned}
f(t) &= z \left[\cos \omega t - \cos (x + \beta)t \right] \\
&- k z^3 \left[\frac{9}{4m} \cos \omega t + \frac{\cos 3 \omega t}{4 (x^2 - 9\omega^2)} + \frac{1}{32x^2} \cos (3x + 3\beta)t \right]
\end{aligned}$$

$$\begin{aligned}
& - \frac{3}{16\omega (x-\omega)} \cos (x-2\omega+\beta)t + \frac{3}{16\omega (x+\omega)} \cos (x+2\omega+\beta)t \\
& - \frac{9}{8m} \cos (x+4\beta)t - \frac{3}{4 (3x+\omega) (x+\omega)} \cos (2x+\omega+2\beta)t \\
& - \frac{3}{4 (3x-\omega) (x-\omega)} \cos (2x-\omega+\beta)t \\
& - \frac{1}{4 (x^2 - 9\omega^2)} \cos (x + \frac{15kz^2}{8x}) t \\
& - \frac{1}{32x^2} \cos (x + \frac{87 kz^2}{4x}) t \\
& - \frac{9}{4(9x^2 - \omega^2)} \cos (x + \frac{21 kz^2}{8x}) t \Big] . \tag{101}
\end{aligned}$$

Actually the number of terms available from computer results is still insufficient to establish Equation (101) as the approximate solution through the first two terms, (k^0 and k^1). The assumptions in regard to the trigonometric relationships are obvious. Evidently one must choose suitable values of ω , x , G and k .

Figure 11 shows the usual numerical solution compared with the evaluation of Equation (101) for the parameters indicated.



11. DIRECT SERIES SOLUTIONS

Direct series solutions may be divided into two categories:

I. Cases when $f(t)$ is not only bounded but meets the requirements of Section 3 for the Laplace transform type of solution.

Section 4 shows an example of the direct series solution for Equation (13) and Figure 1 shows the graphical result for the numerical compared with the analytic result for given numerical constants.

II. Cases where $f(t)$ may or may not be bounded. The transform method of solution does not apply but the derivative computer programs are identical for either Case I or Case II solutions. Case II solutions include all equations under Case I.

The "open-end integral" of Section 3, Equations (4) and (18) may be interpreted as a defined operation which does not involve the Laplace transform. It does however provide a formal solution in the form of a Maclaurin's series.

Define

$$F(p) = p \int_0^\infty f(t) e^{-pt} dt = \sum_{n=0}^q \frac{f^n(0)}{p^n} \quad (102)$$

The integration is to be carried out by parts and is, of course, a series of differentiations. q is the number of derivatives involved, usually less than 60. $f^n(t)$ is the n th derivative of $f(t)$.

Now if $\frac{1}{p^n}$ is defined as in Heaviside's calculus

$$\frac{1}{p^n} \rightarrow \frac{t^n}{n!} \quad (103)$$

and Equation (102) is evaluated as

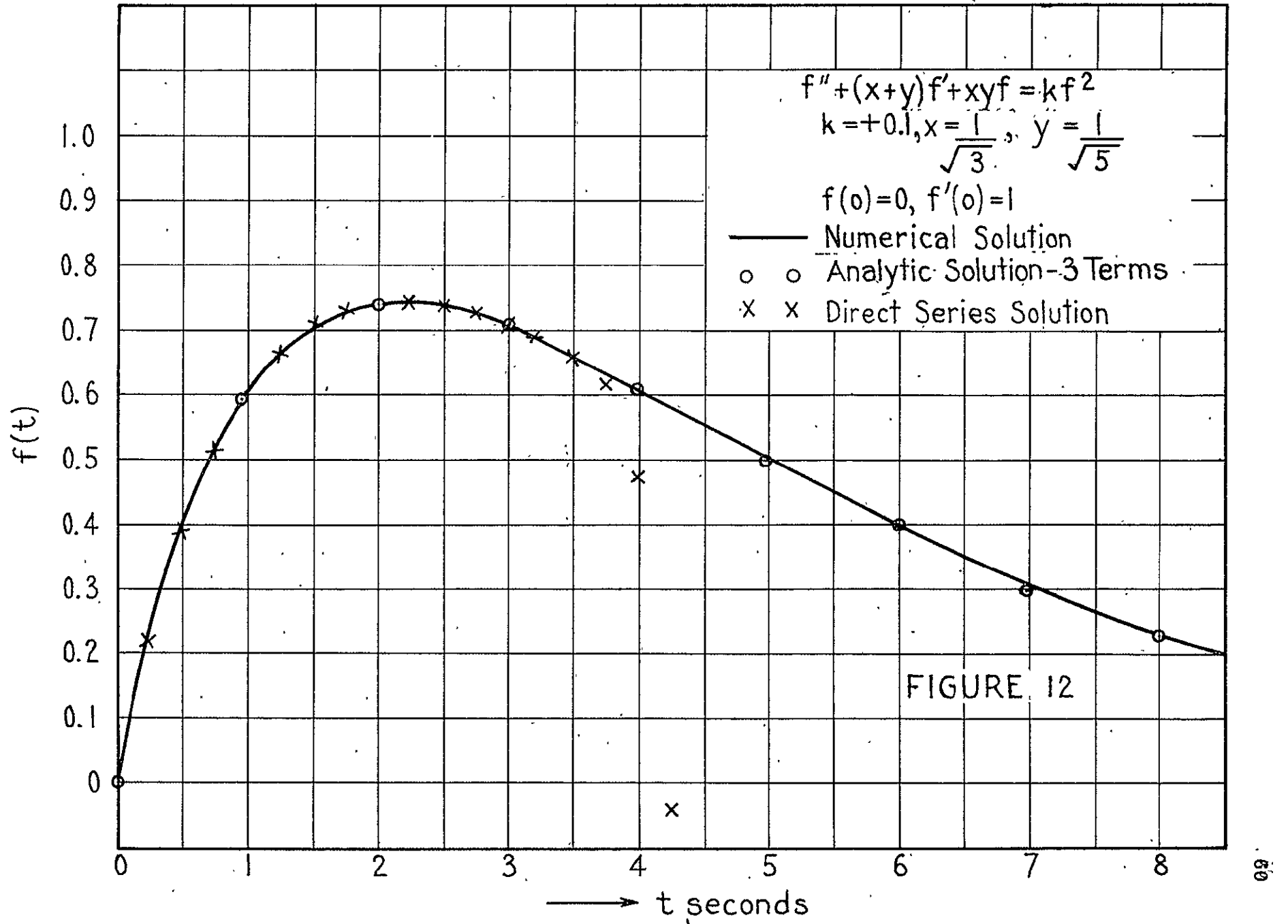
$$f(t) = \sum_{n=0}^q \frac{f^n(0)t^n}{n!} \quad (104)$$

which is Maclaurin's series for $f(t)$ through the q th term. Kaplan¹⁵ has examined equations of the type considered here and has shown that they always converge to an analytic solution for $q \rightarrow \infty$.

Figures 12 and 13 show results for summation and direct series solutions compared with the solution by numerical integration for two equations having Case I solutions. For the same number of derivatives the summation solution maintains accuracy for much larger values of time.

Figures 14 and 15 show comparative results for problems under Case II. No summation solution is available in these two cases.

Kaplan's method of showing convergence also allows an estimate for the remainder of the series after n terms so the accuracy can be estimated in solutions under Case II.



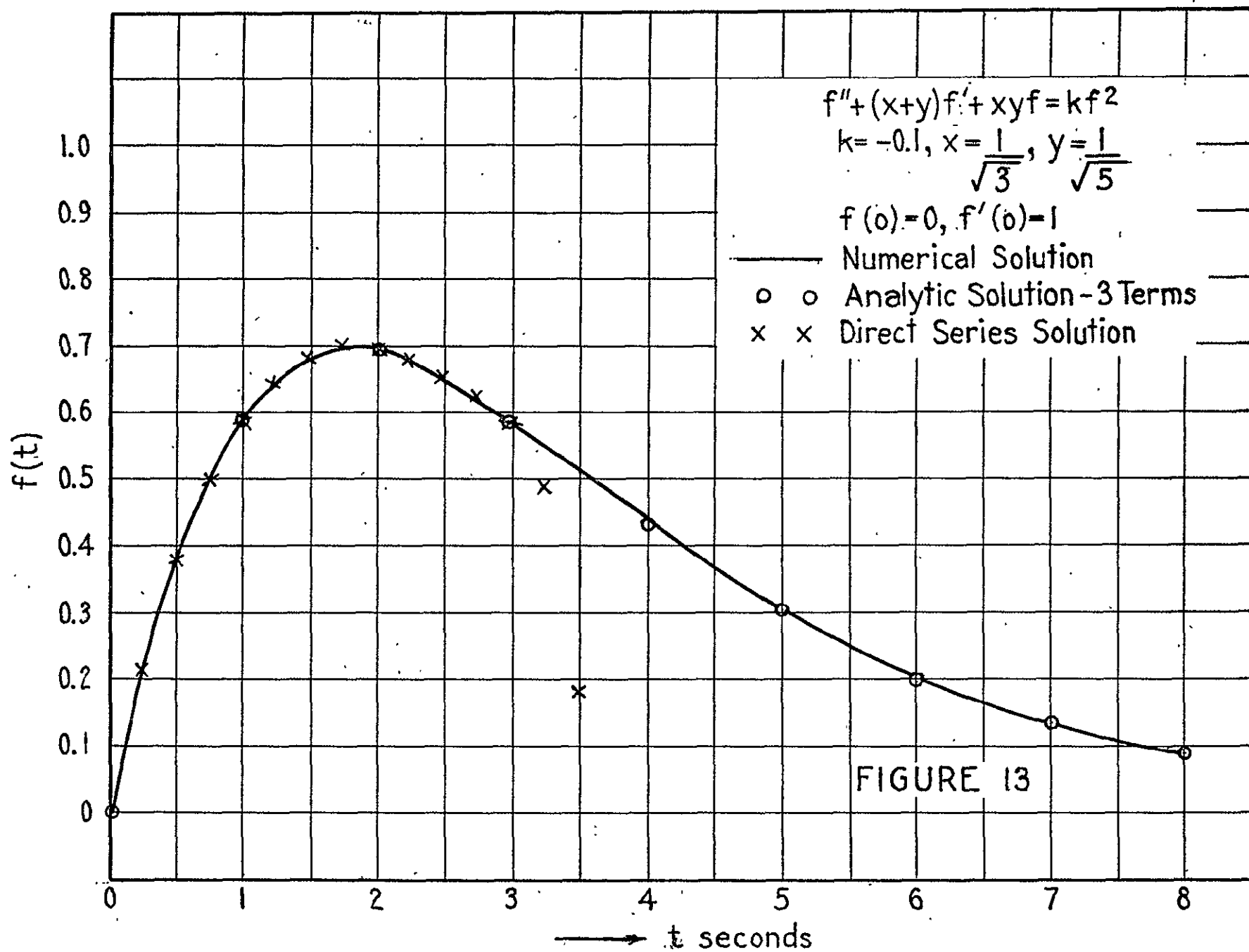
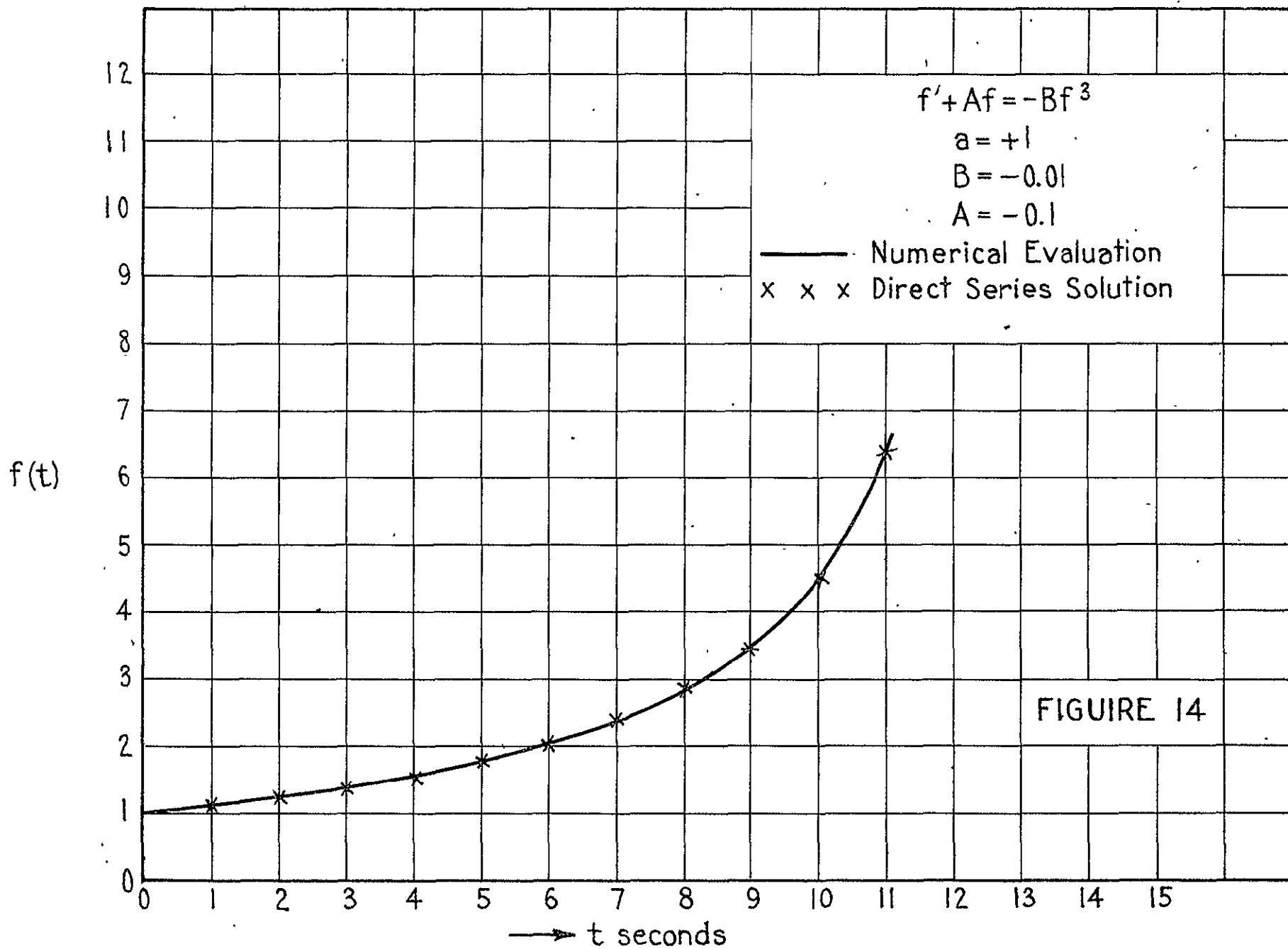


FIGURE 13



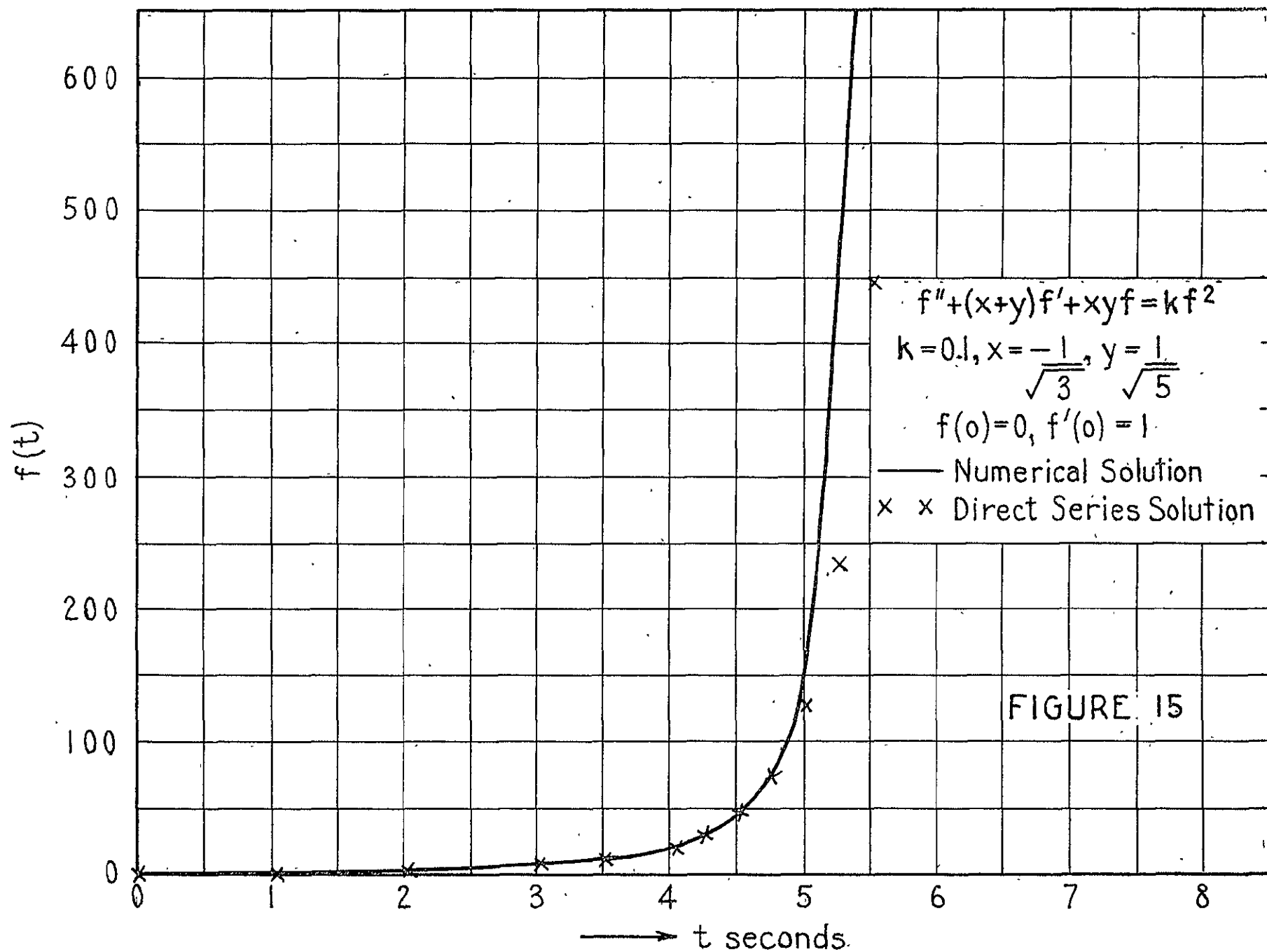


FIGURE 15

12: COMPUTER PROGRAMMING AND OPERATIONS

While some simple non-linear problems may be solved by hand calculations, even relatively simple problems are best solved by the use of a digital computer.

The programming is carried out using the same operations as would be used by hand calculations. Active memory is at a premium so the various operations should be broken down into what might be called irreducible steps. Results are stored on cards, paper tape or preferably magnetic tape.

The numerical coefficients for the terms generated by the process of differentiation cannot be carried in floating point since the last digit of the coefficient is required. In fixed point one must carry coefficients involving some 112 bits or more and sign. A variable word length machine is desirable but not mandatory.

The format for the individual terms should be arranged to carry as many variables as practical in addition to the large coefficient since another problem may require additional variables.

The basic programs required include a program for differentiation, one for summation, one for the numerical evaluation of the transforms and a program for the final evaluation of the time functions.

The program for differentiation involves the input of the individual terms, the differentiation of this term to obtain the next term or terms and a compare and pack routine to store the generated terms in a list in memory. After all terms of a given derivative have been processed the next set of terms forming the next derivative is output to tape and input again to obtain the next derivative. Since each derivative depends upon the tape made from the previous derivative it is essential to check each operation carefully for errors if possible.

The summation operation can be performed using all available derivatives which will go into the computer at one time.

The storage space required is considerably reduced if the derivative terms can be sorted on the basis of the power of the non-linear coefficient involved since this decreases the number of terms to be summed in one run.

If the space required is still more than the available space in memory, additional sorting may be carried out on the powers of the initial conditions.

The actual "summation" computer program is a multiplication operation. If the operational expression

$$\frac{p + A}{(p+B)(p+C)(p+D)}$$

is expanded in memory as a binomial expansion in negative powers of p and the whole list of terms is multiplied successively by $(p+B)$, then $(p+C)$ and finally by $(p+D)$ the final result in memory is $(p+A)$, the numerator.

The numerators of the individual terms are then processed to include the number of roots in each operational expression. This will be called a "numerator tape".

A complete set of the required roots and the initial conditions plus the parameters are then input to an evaluation routine. The numerator tape is input, a term at a time, and for each term the machine evaluates the inverse Laplace Transform and stores the result with a compare and pack routine. The result is the "analytic solution" for the first few terms.

While some operational solutions may be evaluated by hand, and the resulting time function summed into a closed form solution, the usual procedure is to obtain the result as a numerical solution in the independent variable, time. This is then compared with the equivalent numerical solution to determine the range for which the analytic solution is accurate.

The Heaviside shift program for the expansion type solution always holds data in the form ,

$$\frac{(\text{Numerator Polynomial}) t^n e^{mt}}{(\text{Denominator Polynomial})}$$

Except for the numerical coefficients all terms are in algebraic form. This program evaluates the solution faster than the derivative type of solution. Unfortunately the final result must be programmed to evaluate the function in terms of time.

The direct series solution can be evaluated by substituting the numerical value of each constant into the series. Each constant is then multiplied by t^n and added to obtain the sum of the finite series.

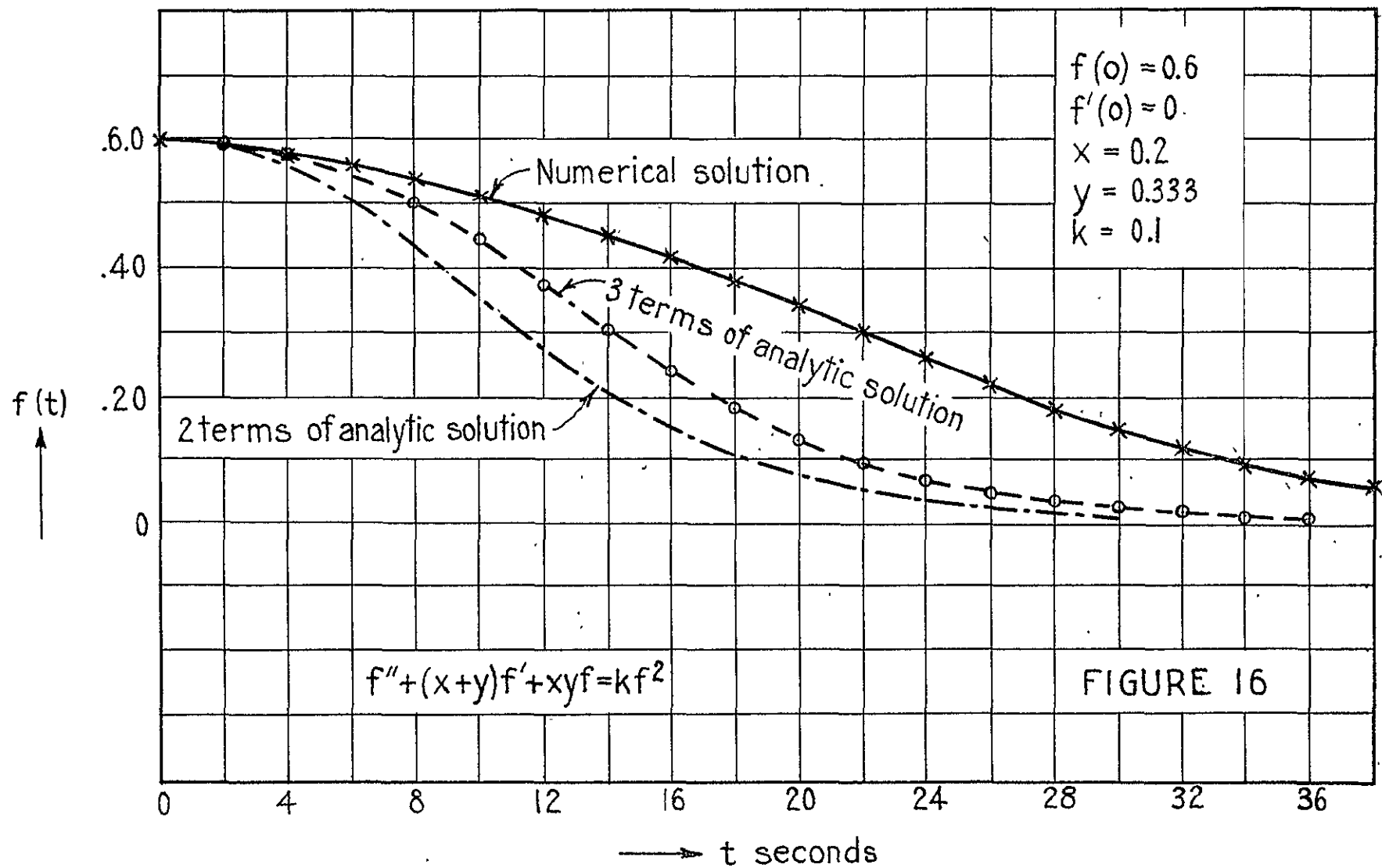
13. CONCLUSIONS

The principal objective of this study was to develop suitable mathematical and programming techniques to obtain analytic solutions for appropriate types of non-linear differential equations using a digital computer.

Two general methods have been developed for the solution of the type of equation considered. For most bounded functions the expansion method is quicker and consequently less expensive in terms of computer time. The derivative method, however, is much more flexible since the solutions may be of bounded or unbounded functions. Summation solutions may be obtained, perhaps in several forms. (Under certain circumstances the results might, for instance, be summed as a Fourier Series.) While the direct series type of derivative solution provides a numerical solution, it is very expensive in terms of computer time and is much less effective than the equivalent summation solution using the same data.

The technique of checking the validity of equations through the use of numerical integration is a very important step in any analytic solution since it provides an engineering type of assurance that the "small parameter," k , is "small enough," at least for that case, and that a sufficient number of terms for convergence is available. Figure 16 shows a case where a very large number of terms would be necessary for reasonable accuracy. It is surprising that the analytic and numerical solutions match as closely as they usually do since both are approximations. The numerical solution has another important function, it points out any peculiarity in the solution which might otherwise have been overlooked.

The appearance of secular terms in non-linear solutions has always been a handicap in hand solutions. Since these terms rarely seem to have any pattern it is pure drudgery to compute them by hand and accuracy is imperative. With computer solutions accuracy must be checked at every step, particularly in input and output, but over-all accuracy is more or



less assured. One may obtain a sufficient number of secular terms to approximate the function or one may be able to recognize the pattern and assimilate the terms into available functions. In either case the computer is an invaluable aid.

This study may be regarded as of an experimental character. Much is yet to be done. No attempt has been made to make a thorough study of any of the equations considered. In many instances the same equation may have many different forms depending upon the value of the parameters and the initial conditions. Since initial conditions frequently appear as a part of the argument of the functions involved, the resulting solution may be quite complicated in structure.

One may well ask why analytic solutions are of interest since digital computers may easily be programmed to provide numerical solutions. The answer is, of course, that numerical solutions can no more take the place of analytic solutions for non-linear work than they could for linear work. One analytic solution is likely to provide more understanding of a given phenomenon than many numerical solutions.

Many books and papers have been written on the theory of non-linear differential equations, but the student finds discouragingly few actual solutions. While the techniques which have been developed during this study are merely an entering wedge and appropriate for a particular type of equation, they point to the possibility of using a computer for analytic solutions in other non-linear areas and indicate that any real future progress in non-linear mathematics is likely to be closely associated with computer work.

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APPENDIX

$$\begin{aligned}
f_4 = k^3 b^4 & \left[-\frac{2a^7 e^{-xt}(e^{-3xt}-1)}{3x^3} + \frac{2a^7 e^{-xt}(e^{-xt}-1)}{x^3} + \frac{2a^7 e^{-xt}(e^{-2xt}-1)}{x^3} \right. \\
& - \frac{4a^7 e^{-xt}(e^{-xt}-1)}{x^3} - \frac{8a^6 c e^{-xt}(e^{-(2x+y)t}-1)}{y(2x+y)(x+y)} + \frac{8a^6 c e^{-xt}(e^{-xt}-1)}{xy(x+y)} \\
& + \frac{4a^6 c e^{-xt}(e^{-2xt}-1)}{x^2 y} - \frac{12a^6 c e^{-xt}(e^{-xt}-1)}{x^2 y} + \frac{2a^5 c^2 e^{-xt}(e^{-(x+2y)t}-1)}{y(x-2y)(x+2y)} \\
& - \frac{2a^5 c^2 e^{-xt}(e^{-xt}-1)}{xy(x-2y)} - \frac{2a^5 c^2 e^{-xt}(e^{-2xt}-1)}{x^2(x-2y)} + \frac{4a^5 c^2 e^{-xt}(e^{-xt}-1)}{x^2(x-2y)} \\
& - \frac{2a^6 c e^{-xt}(e^{-3xt}-1)}{3x^2(2x-y)} + \frac{2a^6 c e^{-xt}(e^{-xt}-1)}{x^2(2x-y)} + \frac{4a^6 c e^{-xt}(e^{-(x+y)t}-1)}{y(2x-y)(x+y)} \\
& - \frac{4a^6 c e^{-xt}(e^{-xt}-1)}{xy(2x-y)} - \frac{8a^5 c^2 e^{-xt}(e^{-(2x+y)t}-1)}{x(2x+y)(x+y)} + \frac{8a^5 c^2 e^{-xt}(e^{-xt}-1)}{x^2(x+2y)} \\
& + \frac{16a^5 c^2 e^{-xt}(e^{-(x+y)t}-1)}{xy(x+y)} - \frac{8a^5 c^2 e^{-xt}(e^{-xt}-1)}{x^2 y} \\
& - \frac{2a^4 c^3 e^{-xt}(e^{-(x+2y)t}-1)}{y^2(x+2y)} + \frac{2a^4 c^3 e^{-xt}(e^{-xt}-1)}{xy^2} \\
& + \frac{4a^4 c^3 e^{-xt}(e^{-(x+y)t}-1)}{y^2(x+y)} - \frac{4a^4 c^3 e^{-xt}(e^{-xt}-1)}{xy^2} \\
& - \frac{4a^6 c e^{-xt}(e^{-(2x+y)t}-1)}{x(2x+y)(x+y)} + \frac{4a^6 c e^{-xt}(e^{-xt}-1)}{x^2(x+y)} \\
& + \frac{4a^6 c e^{-xt}(e^{-(x+y)t}-1)}{xy(x+y)} - \frac{4a^5 c^2 e^{-xt}(e^{-(x+2y)t}-1)}{y^2(x+2y)} \\
& + \frac{4a^5 c^2 e^{-xt}(e^{-xt}-1)}{xy^2} + \frac{8a^5 c^2 e^{-xt}(e^{-(x+y)t}-1)}{y^2(x+y)} - \frac{8a^5 c^2 e^{-xt}(e^{-xt}-1)}{xy^2} \Big]
\end{aligned}$$

Cont.

f_4 Continued

$$\begin{aligned}
& + k^3 b^4 \left[- \frac{4a^4 c^3 e^{-xt} (e^{-3yt} - 1)}{3y(x-2y)(x-3y)} + \frac{4a^4 c^3 e^{-xt} (e^{-xt} - 1)}{x(x-2y)(x-3y)} - \frac{4a^4 c^3 e^{-xt} (e^{-(x+y)t} - 1)}{y(x-2y)(x+y)} \right. \\
& + \frac{4a^4 c^3 e^{-xt} (e^{-xt} - 1)}{xy(x-2y)} - \frac{4a^5 c^2 e^{-xt} (e^{-(2x+y)t} - 1)}{(2x+y)(2x-y)(x+y)} + \frac{4a^5 c^2 e^{-xt} (e^{-xt} - 1)}{x(2x-y)(x+y)} \\
& - \frac{2a^5 c^2 e^{-xt} (e^{-2yt} - 1)}{y(2x-y)(x-2y)} + \frac{4a^5 c^2 e^{-xt} (e^{-xt} - 1)}{x(2x-y)(x-2y)} - \frac{8a^4 c^3 e^{-xt} (e^{-(x+2y)t} - 1)}{xy(x+2y)} \\
& + \frac{4a^4 c^3 e^{-xt} (e^{-xt} - 1)}{x^2 y} - \frac{4a^4 c^3 e^{-xt} (e^{-2yt} - 1)}{xy(x-2y)} + \frac{8a^4 c^3 e^{-xt} (e^{-xt} - 1)}{x^2 (x-2y)} \\
& + \frac{4a^3 c^4 e^{-xt} (e^{-3yt} - 1)}{3y^2 (x-3y)} - \frac{4a^3 c^4 e^{-xt} (e^{-xt} - 1)}{xy(x-3y)} - \frac{4a^3 c^4 e^{-xt} (e^{-2yt} - 1)}{y^2 (x-2y)} \\
& + \frac{4a^3 c^4 e^{-xt} (e^{-xt} - 1)}{xy(x-2y)} - \frac{4a^6 c e^{-xt} (e^{-3xt} - 1)}{3x^2 (3x-y)} + \frac{4a^6 c e^{-xt} (e^{-yt} - 1)}{xy(3x-y)} \\
& + \frac{2a^6 c e^{-xt} (e^{-2xt} - 1)}{x^2 (2x-y)} - \frac{4a^6 c e^{-xt} (e^{-yt} - 1)}{xy(2x-y)} - \frac{4a^5 c^2 e^{-xt} (e^{-2xt} - 1)}{xy(2x+y)} \\
& + \frac{4a^5 c^2 e^{-xt} (e^{-yt} - 1)}{xy^2} + \frac{4a^5 c^2 e^{-xt} (e^{-2xt} - 1)}{xy(2x-y)} - \frac{12a^5 c^2 e^{-xt} (e^{-yt} - 1)}{y^2 (2x-y)} \\
& + \frac{4a^4 c^3 e^{-xt} (e^{-(x+2y)t} - 1)}{(x+2y)(x-2y)(x+y)} - \frac{4a^4 c^3 e^{-xt} (e^{-yt} - 1)}{y(x-2y)(x+y)} - \frac{2a^4 c^3 e^{-xt} (e^{-2xt} - 1)}{x(2x-y)(x-2y)} \\
& + \frac{8a^4 c^3 e^{-xt} (e^{-yt} - 1)}{y(2x-y)(x-2y)} - \frac{4a^5 c^2 e^{-xt} (e^{-3xt} - 1)}{3x(3x-y)(2x-y)} + \frac{4a^5 c^2 e^{-xt} (e^{-yt} - 1)}{y(2x-y)(3x-y)} \\
& + \frac{8a^5 c^2 e^{-xt} (e^{-(x+y)t} - 1)}{x(2x-y)(x+y)} - \frac{4a^5 c^2 e^{-xt} (e^{-yt} - 1)}{xy(2x-y)} \\
& \left. - \frac{4a^4 c^3 e^{-xt} (e^{-(2x+y)t} - 1)}{x^2 (2x+y)} + \frac{4a^4 c^3 e^{-xt} (e^{-yt} - 1)}{x^2 y} \right]
\end{aligned}$$

Cont.

f₄ Continued

$$\begin{aligned}
& + k^3 b^4 \left[\begin{aligned}
& + \frac{8a^4 c^3 e^{-xt} (e^{-(x+y)t} - 1)}{x^2 (x+y)} - \frac{8a^4 c^3 e^{-xt} (e^{-yt} - 1)}{x^2 y} \\
& - \frac{4a^3 c^4 e^{-xt} (e^{-(x+2y)t} - 1)}{y(x+2y)(x+y)} + \frac{4a^3 c^4 e^{-xt} (e^{-yt} - 1)}{y^2 (x+y)} \\
& + \frac{4a^3 c^4 e^{-xt} (e^{-(x+y)t} - 1)}{xy(x+y)} - \frac{12a^3 c^4 e^{-xt} (e^{-yt} - 1)}{xy^2} \\
& - \frac{2a^5 c^2 e^{-xt} (e^{-(2x+y)t} - 1)}{x^2 (2x+y)} + \frac{2a^5 c^2 e^{-xt} (e^{-yt} - 1)}{x^2 y} \\
& + \frac{4a^5 c^2 e^{-xt} (e^{-(x+y)t} - 1)}{x^2 (x+y)} - \frac{4a^5 c^2 e^{-xt} (e^{-yt} - 1)}{x^2 y} \\
& - \frac{4a^4 c^3 e^{-xt} (e^{-(x+2y)t} - 1)}{y(x+2y)(x+y)} + \frac{8a^4 c^3 e^{-xt} (e^{-yt} - 1)}{y^2 (x+y)} \\
& + \frac{8a^4 c^3 e^{-xt} (e^{-(x+y)t} - 1)}{xy(x+y)} - \frac{16a^4 c^3 e^{-xt} (e^{-yt} - 1)}{xy^2} + \frac{2a^3 c^4 e^{-xt} (e^{-3yt} - 1)}{3y^2 (x-2y)} \\
& - \frac{2a^3 c^4 e^{-xt} (e^{-yt} - 1)}{y^2 (x-2y)} - \frac{4a^3 c^4 e^{-xt} (e^{-(x+y)t} - 1)}{x(x-2y)(x+y)} + \frac{12a^3 c^4 e^{-xt} (e^{-yt} - 1)}{xy(x-2y)} \\
& - \frac{2a^4 c^3 e^{-xt} (e^{-(2x+y)t} - 1)}{x(2x-y)(2x+y)} + \frac{2a^4 c^3 e^{-xt} (e^{-yt} - 1)}{xy(2x-y)} - \frac{2a^4 c^3 e^{-xt} (e^{-2yt} - 1)}{y^2 (2x-y)} \\
& - \frac{4a^4 c^3 e^{-xt} (e^{-yt} - 1)}{y^2 (2x-y)} - \frac{8a^3 c^4 e^{-xt} (e^{-(x+2y)t} - 1)}{x(x+2y)(x+y)} + \frac{8a^3 c^4 e^{-xt} (e^{-yt} - 1)}{xy(x+y)} \\
& + \frac{8a^3 c^4 e^{-xt} (e^{-2yt} - 1)}{xy^2} - \frac{2a^2 c^5 e^{-xt} (e^{-3yt} - 1)}{3y^3} + \frac{2a^2 c^5 e^{-xt} (e^{-yt} - 1)}{y^3} \\
& + \frac{2a^2 c^5 e^{-xt} (e^{-2yt} - 1)}{y^3} - \frac{4a^2 c^5 e^{-xt} (e^{-yt} - 1)}{y^3} - \frac{2a^6 c e^{-xt} (e^{-(2x+y)t} - 1)}{x^2 (2x+y)}
\end{aligned} \right]
\end{aligned}$$

Cont.

f₄ Continued

$$\begin{aligned}
& + k^3 b^4 \left[\begin{aligned}
& + \frac{2a^6 c e^{-xt} (e^{-yt} - 1)}{x^2 y} + \frac{4a^6 c e^{-xt} (e^{-(x+y)t} - 1)}{x^2 (x+y)} - \frac{4a^6 c e^{-xt} (e^{-yt} - 1)}{x^2 y} \\
& - \frac{8a^5 c^2 e^{-xt} (e^{-(x+2y)t} - 1)}{y(x+y)(x+2y)} + \frac{8a^5 c^2 e^{-xt} (e^{-yt} - 1)}{y^2 (x+y)} - \frac{12a^5 c^2 e^{-xt} (e^{-yt} - 1)}{xy^2} \\
& + \frac{2a^4 c^3 e^{-xt} (e^{-3yt} - 1)}{3y^2 (x-2y)} - \frac{2a^4 c^3 e^{-xt} (e^{-yt} - 1)}{y^2 (x-2y)} - \frac{4a^4 c^3 e^{-xt} (e^{-(x+y)t} - 1)}{x(x-2y)(x+y)} \\
& + \frac{4a^4 c^3 e^{-xt} (e^{-yt} - 1)}{xy(x-2y)} - \frac{2a^5 c^2 e^{-xt} (e^{-(2x+y)t} - 1)}{x(2x-y)(2x+y)} + \frac{2a^5 c^2 e^{-xt} (e^{-yt} - 1)}{xy(2x-y)} \\
& + \frac{2a^5 c^2 e^{-xt} (e^{-2yt} - 1)}{y^2 (2x-y)} - \frac{8a^4 c^3 e^{-xt} (e^{-(x+2y)t} - 1)}{x(x+2y)(x+y)} + \frac{8a^4 c^3 e^{-xt} (e^{-yt} - 1)}{xy(x+y)} \\
& + \frac{4a^4 c^3 e^{-xt} (e^{-2yt} - 1)}{xy^2} - \frac{2a^3 c^4 e^{-xt} (e^{-3yt} - 1)}{3y^3} + \frac{2a^3 c^4 e^{-xt} (e^{-yt} - 1)}{y^3} \\
& + \frac{2a^3 c^4 e^{-xt} (e^{-2yt} - 1)}{y^3} - \frac{4a^3 c^4 e^{-xt} (e^{-yt} - 1)}{y^3} - \frac{4a^5 c^2 e^{-xt} (e^{-(x+2y)t} - 1)}{x(x+y)(x+2y)} \\
& + \frac{4a^5 c^2 e^{-xt} (e^{-yt} - 1)}{xy(x+y)} + \frac{2a^5 c^2 e^{-xt} (e^{-2yt} - 1)}{xy^2} - \frac{4a^4 c^3 e^{-xt} (e^{-3yt} - 1)}{3y^3} \\
& + \frac{4a^4 c^3 e^{-xt} (e^{-yt} - 1)}{y^3} + \frac{4a^4 c^3 e^{-xt} (e^{-2yt} - 1)}{y^3} - \frac{8a^4 c^3 e^{-xt} (e^{-yt} - 1)}{y^3} \\
& + \frac{4a^3 c^4 e^{-xt} (e^{(x-4y)t} - 1)}{(x-3y)(x-4y)(x-2y)} + \frac{4a^3 c^4 e^{-xt} (e^{-yt} - 1)}{y(x-2y)(x-3y)} + \frac{4a^3 c^4 e^{-xt} (e^{-yt} - 1)}{y^2 (x-2y)} \\
& - \frac{4a^4 c^3 e^{-xt} (e^{-(x+2y)t} - 1)}{(2x-y)(x+2y)(x+y)} + \frac{4a^4 c^3 e^{-xt} (e^{-yt} - 1)}{y(2x-y)(x+y)} \\
& + \frac{4a^4 c^3 e^{-xt} (e^{(x-3y)t} - 1)}{(2x-y)(x-3y)(x-2y)} - \frac{4a^3 c^4 e^{-xt} (e^{-3yt} - 1)}{3xy^2} + \frac{4a^3 c^4 e^{-xt} (e^{-yt} - 1)}{xy^2}
\end{aligned} \right]
\end{aligned}$$

Cont.

f₄ Continued

$$\begin{aligned}
& + k^3 b^4 \left[\begin{aligned}
& + \frac{8a^3 c^4 e^{-xt} (e^{(x-3y)t} - 1)}{x(x-2y)(x-3y)} - \frac{4a^2 c^5 e^{-xt} (e^{(x-4y)t} - 1)}{y(x-3y)(x-4y)} \\
& - \frac{4a^2 c^5 e^{-xt} (e^{-yt} - 1)}{y^2(x-3y)} + \frac{4a^2 c^5 e^{-xt} (e^{(x-3y)t} - 1)}{y(x-2y)(x-3y)} + \frac{4a^2 c^5 e^{-xt} (e^{-yt} - 1)}{y^2(x-2y)} \\
& - \frac{4a^5 c^2 e^{-xt} (e^{-(2x+y)t} - 1)}{x(3x-y)(2x+y)} - \frac{4a^5 c^2 e^{-xt} (e^{(x-2y)t} - 1)}{x(3x-y)(x-2y)} \\
& + \frac{4a^5 c^2 e^{-xt} (e^{(x-2y)t} - 1)}{x(2x-y)(x-2y)} - \frac{4a^4 c^3 e^{-xt} (e^{(x-2y)t} - 1)}{xy(x-2y)} \\
& + \frac{8a^4 c^3 e^{-xt} (e^{-(x+y)t} - 1)}{y(2x-y)(x+y)} + \frac{8a^4 c^3 e^{-xt} (e^{(x-2y)t} - 1)}{y(2x-y)(x-2y)} \\
& + \frac{4a^3 c^4 e^{-xt} (e^{-3yt} - 1)}{3y(x-2y)(x+y)} + \frac{4a^3 c^4 e^{-xt} (e^{(x-2y)t} - 1)}{(x-2y)^2(x+y)} \\
& - \frac{4a^3 c^4 e^{-xt} (e^{-(x+y)t} - 1)}{(x-2y)(2x-y)(x+y)} - \frac{4a^3 c^4 e^{-xt} (e^{(x-2y)t} - 1)}{(2x-y)(x-2y)^2} \\
& - \frac{4a^4 c^3 e^{-xt} (e^{-(2x+y)t} - 1)}{(3x-y)(2x-y)(2x+y)} - \frac{4a^4 c^3 e^{-xt} (e^{(x-2y)t} - 1)}{(x-2y)(3x-y)(2x-y)} \\
& + \frac{2a^4 c^3 e^{-xt} (e^{-2yt} - 1)}{xy(2x-y)} + \frac{4a^4 c^3 e^{-xt} (e^{(x-2y)t} - 1)}{x(2x-y)(x-2y)} \\
& - \frac{4a^3 c^4 e^{-xt} (e^{-(x+2y)t} - 1)}{x^2(x+2y)} - \frac{4a^3 c^4 e^{-xt} (e^{(x-2y)t} - 1)}{x^2(x-2y)} \\
& + \frac{4a^3 c^4 e^{-xt} (e^{-2yt} - 1)}{x^2 y} + \frac{8a^3 c^4 e^{-xt} (e^{(x-2y)t} - 1)}{x^2(x-2y)} - \frac{4a^2 c^5 e^{-xt} (e^{-3yt} - 1)}{3y^2(x+y)} \\
& - \frac{4a^2 c^5 e^{-xt} (e^{(x-2y)t} - 1)}{y(x-2y)(x+y)} + \frac{2a^2 c^5 e^{-xt} (e^{-2yt} - 1)}{xy^2}
\end{aligned} \right]
\end{aligned}$$

Cont.

f_4 Continued

$+ k^3 b^4$

$$\begin{aligned}
 & + \frac{12a^2c^5e^{-xt}(e^{(x-2y)t}-1)}{xy(x-2y)} - \frac{2a^4c^3e^{-xt}(e^{-(x-2y)t}-1)}{x^2(x+2y)} \\
 & - \frac{2a^4c^3e^{-xt}(e^{(x-2y)t}-1)}{x^2(x-2y)} + \frac{2a^4c^3e^{-xt}(e^{-2yt}-1)}{x^2y} \\
 & + \frac{4a^4c^3e^{-xt}(e^{(x-2y)t}-1)}{x^2(x-2y)} - \frac{8a^3c^4e^{-xt}(e^{-3yt}-1)}{3y^2(x+y)} \\
 & - \frac{8a^3c^4e^{-xt}(e^{(x-2y)t}-1)}{y(x-2y)(x+y)} + \frac{8a^3c^4e^{-xt}(e^{(x-2y)t}-1)}{xy(x-2y)} \\
 & - \frac{2a^2c^5e^{-xt}(e^{(x-4y)t}-1)}{y(x-4y)(x-2y)} + \frac{2a^2c^5e^{-xt}(e^{(x-2y)t}-1)}{y(x-2y)^2} \\
 & - \frac{2a^2c^5e^{-xt}(e^{-2yt}-1)}{xy(x-2y)} - \frac{4a^2c^5e^{-xt}(e^{(x-2y)t}-1)}{x(x-2y)^2} \\
 & - \frac{2a^3c^4e^{-xt}(e^{-(x+2y)t}-1)}{x(2x-y)(x+2y)} - \frac{2a^3c^4e^{-xt}(e^{(x-2y)t}-1)}{x(2x-y)(x-2y)} \\
 & - \frac{4a^3c^4e^{-xt}(e^{(x-3y)t}-1)}{y(2x-y)(x-3y)} + \frac{4a^3c^4e^{-xt}(e^{(x-2y)t}-1)}{y(2x-y)(x-2y)} \\
 & - \frac{8a^2c^5e^{-xt}(e^{-3yt}-1)}{3xy(x+y)} - \frac{8a^2c^5e^{-xt}(e^{(x-2y)t}-1)}{x(x-2y)(x+y)} \\
 & - \frac{8a^2c^5e^{-xt}(e^{-(x+3y)t}-1)}{xy(x-3y)} + \frac{2ac^6e^{-xt}(e^{(x-4y)t}-1)}{y^2(x-4y)} \\
 & - \frac{2ac^6e^{-xt}(e^{(x-2y)t}-1)}{y^2(x-2y)} - \frac{4ac^6e^{-xt}(e^{(x-3y)t}-1)}{y^2(x-3y)} \\
 & + \frac{4ac^6e^{-xt}(e^{(x-2y)t}-1)}{y^2(x-2y)}
 \end{aligned}$$

Cont.

f₄ Continued

$$\begin{aligned}
& + k^3 b^4 \left[\frac{4a^5 c^2 e^{-yt} (e^{(y-2x)t} - 1)}{x(2x-y)(x+y)} + \frac{2a^5 c^2 e^{-yt} (e^{-2xt} - 1)}{x^2 y} \right. \\
& - \frac{4a^4 c^3 e^{-yt} (e^{-(2x+y)t} - 1)}{y^2 (2x+y)} + \frac{4a^4 c^3 e^{-yt} (e^{(y-2x)t} - 1)}{y^2 (2x-y)} \\
& + \frac{4a^4 c^3 e^{-yt} (e^{-2xt} - 1)}{xy^2} - \frac{8a^4 c^3 e^{-yt} (e^{(y-2x)t} - 1)}{y^2 (2x-y)} \\
& - \frac{4a^3 c^4 e^{-yt} (e^{-(x+2y)t} - 1)}{(x-3y)(x+2y)(x-2y)} + \frac{4a^3 c^4 e^{-yt} (e^{(y-2x)t} - 1)}{(2x-y)(x-3y)(x-2y)} \\
& - \frac{2a^3 c^4 e^{-yt} (e^{-2xt} - 1)}{xy(x-2y)} + \frac{4a^3 c^4 e^{-yt} (e^{(y-2x)t} - 1)}{y(2x-y)(x-2y)} \\
& + \frac{4a^4 c^3 e^{-yt} (e^{(y-2x)t} - 1)}{(x+y)(2x-y)^2} - \frac{4a^4 c^3 e^{-yt} (e^{-(x+y)t} - 1)}{(2x-y)(x-2y)(x+y)} \\
& + \frac{4a^4 c^3 e^{-yt} (e^{(y-2x)t} - 1)}{(2x-y)^2 (x-2y)} - \frac{8a^3 c^4 e^{-yt} (e^{-(2x+y)t} - 1)}{xy(2x+y)} \\
& - \frac{4a^3 c^4 e^{-yt} (e^{(y-2x)t} - 1)}{xy(2x-y)} - \frac{8a^3 c^4 e^{-yt} (e^{-(x+y)t} - 1)}{x(x-2y)(x+y)} \\
& + \frac{8a^3 c^4 e^{-yt} (e^{(y-2x)t} - 1)}{x(x-2y)(2x-y)} + \frac{4a^2 c^5 e^{-yt} (e^{-(x+2y)t} - 1)}{y(x-3y)(x+2y)} \\
& - \frac{4a^2 c^5 e^{-yt} (e^{(y-2x)t} - 1)}{y(2x-y)(x-3y)} - \frac{8a^2 c^5 e^{-yt} (e^{-(x+y)t} - 1)}{y(x+y)(x-2y)} \\
& + \frac{4a^2 c^5 e^{-yt} (e^{(y-2x)t} - 1)}{y(2x-y)(x-2y)} - \frac{4a^5 c^2 e^{-yt} (e^{(y-4x)t} - 1)}{x(4x-y)(3x-y)} \\
& + \frac{4a^5 c^2 e^{-yt} (e^{-xt} - 1)}{x^2 (3x-y)} + \frac{4a^5 c^2 e^{-yt} (e^{(y-3x)t} - 1)}{x(3x-y)(2x-y)} - \frac{4a^5 c^2 e^{-yt} (e^{-xt} - 1)}{x^2 (2x-y)}
\end{aligned}$$

Cont.

f₄ Continued+ k³b⁴

$$\begin{aligned}
& - \frac{4a^4c^3e^{-yt}(e^{-3xt}-1)}{3x^2y} + \frac{4a^4c^3e^{-yt}(e^{-xt}-1)}{x^2y} + \frac{8a^4c^3e^{-yt}(e^{(y-3x)t}-1)}{y(3x-y)(2x-y)} \\
& - \frac{12a^4c^3e^{-yt}(e^{-xt}-1)}{xy(2x-y)} + \frac{4a^3c^4e^{-yt}(e^{-(2x+y)t}-1)}{(x-2y)(2x+y)(x+y)} - \frac{4a^3c^4e^{-yt}(e^{-xt}-1)}{x(x-2y)(x+y)} \\
& - \frac{4a^3c^4e^{-yt}(e^{(y-3x)t}-1)}{(3x-y)(2x-y)(x-2y)} + \frac{8a^3c^4e^{-yt}(e^{-xt}-1)}{x(2x-y)(x-2y)} \\
& - \frac{4a^4c^3e^{-yt}(e^{(y-4x)t}-1)}{(4x-y)(3x-y)(2x-y)} + \frac{4a^4c^3e^{-yt}(e^{-xt}-1)}{x(3x-y)(2x-y)} + \frac{4a^4c^3e^{-yt}(e^{-2xt}-1)}{x^2(2x-y)} \\
& - \frac{4a^4c^3e^{-yt}(e^{-xt}-1)}{x^2(2x-y)} - \frac{4a^3c^4e^{-yt}(e^{-3xt}-1)}{3x^3} + \frac{4a^3c^4e^{-yt}(e^{-xt}-1)}{x^3} \\
& + \frac{4a^3c^4e^{-yt}(e^{-2xt}-1)}{x^3} - \frac{8a^3c^4e^{-yt}(e^{-xt}-1)}{x^3} - \frac{4a^2c^5e^{-yt}(e^{-(2x+y)t}-1)}{y(2x+y)(x+y)} \\
& + \frac{4a^2c^5e^{-yt}(e^{-xt}-1)}{xy(x+y)} + \frac{2a^2c^5e^{-yt}(e^{-2xt}-1)}{x^2y} - \frac{12a^2c^5e^{-yt}(e^{-xt}-1)}{x^2y} \\
& - \frac{2a^4c^3e^{-yt}(e^{-3xt}-1)}{3x^3} + \frac{2a^4c^3e^{-yt}(e^{-xt}-1)}{x^3} + \frac{2a^4c^3e^{-yt}(e^{-2xt}-1)}{x^3} \\
& - \frac{4a^4c^3e^{-yt}(e^{-xt}-1)}{x^3} - \frac{8a^3c^4e^{-yt}(e^{-(2x+y)t}-1)}{y(2x+y)(x+y)} + \frac{8a^3c^4e^{-yt}(e^{-xt}-1)}{xy(x+y)} \\
& + \frac{4a^3c^4e^{-yt}(e^{-2xt}-1)}{x^2y} - \frac{16a^3c^4e^{-yt}(e^{-xt}-1)}{x^2y} + \frac{2a^2c^5e^{-yt}(e^{-(x+2y)t}-1)}{y(x-2y)(x+2y)} \\
& + \frac{2a^2c^5e^{-yt}(e^{-xt}-1)}{xy(x-2y)} - \frac{2a^2c^5e^{-yt}(e^{-2xt}-1)}{x^2(x-2y)} + \frac{12a^2c^5e^{-yt}(e^{-xt}-1)}{x^2(x-2y)} \\
& - \frac{2a^3c^4e^{-yt}(e^{-3xt}-1)}{3x^2(2x-y)} + \frac{2a^3c^4e^{-yt}(e^{-xt}-1)}{x^2(2x-y)} + \frac{4a^3c^4e^{-yt}(e^{-(x+y)t}-1)}{y(2x-y)(x+y)}
\end{aligned}$$

Cont.

f_4 Continued $+ k^3 b^4$

$$\begin{aligned}
& - \frac{4a^3 c^4 e^{-yt} (e^{-xt} - 1)}{xy(2x-y)} - \frac{8a^2 c^5 e^{-yt} (e^{-(2x+y)t} - 1)}{x(2x+y)(x+y)} + \frac{8a^2 c^5 e^{-yt} (e^{-xt} - 1)}{x^2(x+y)} \\
& + \frac{16a^2 c^5 e^{-yt} (e^{-(x+y)t} - 1)}{xy(x+y)} - \frac{2ac^6 e^{-yt} (e^{-(x+2y)t} - 1)}{y^2(x+2y)} \\
& + \frac{2ac^6 e^{-yt} (e^{-xt} - 1)}{xy^2} + \frac{4ac^6 e^{-yt} (e^{-(x+y)t} - 1)}{y^2(x+y)} - \frac{4ac^6 e^{-yt} (e^{-xt} - 1)}{xy^2} \\
& - \frac{2a^5 c^2 e^{-yt} (e^{-3xt} - 1)}{3x^3} + \frac{2a^5 c^2 e^{-yt} (e^{-xt} - 1)}{x^3} + \frac{2a^5 c^2 e^{-yt} (e^{-2xt} - 1)}{x^3} \\
& - \frac{4a^5 c^2 e^{-yt} (e^{-xt} - 1)}{x^3} - \frac{8a^4 c^3 e^{-yt} (e^{-(2x+y)t} - 1)}{y(2x+y)(x+y)} + \frac{8a^4 c^3 e^{-yt} (e^{-xt} - 1)}{xy(x+y)} \\
& - \frac{12a^4 c^3 e^{-yt} (e^{-xt} - 1)}{x^2 y} + \frac{2a^3 c^4 e^{-yt} (e^{-(x+2y)t} - 1)}{y(x-2y)(x+2y)} - \frac{2a^3 c^4 e^{-yt} (e^{-xt} - 1)}{xy(x-2y)} \\
& - \frac{2a^3 c^4 e^{-yt} (e^{-2xt} - 1)}{x^2(x-2y)} + \frac{4a^3 c^4 e^{-yt} (e^{-xt} - 1)}{x^2(x-2y)} - \frac{2a^4 c^3 e^{-yt} (e^{-3xt} - 1)}{3x^2(2x-y)} \\
& + \frac{2a^4 c^3 e^{-yt} (e^{-xt} - 1)}{x^2(2x-y)} + \frac{4a^4 c^3 e^{-yt} (e^{-(x+y)t} - 1)}{y(2x-y)(x+y)} \\
& - \frac{8a^3 c^4 e^{-yt} (e^{-(2x+y)t} - 1)}{x(2x+y)(x+y)} + \frac{8a^3 c^4 e^{-yt} (e^{-xt} - 1)}{x^2(x+y)} \\
& + \frac{8a^3 c^4 e^{-yt} (e^{-(x+y)t} - 1)}{xy(x+y)} - \frac{2a^2 c^5 e^{-yt} (e^{-(x+2y)t} - 1)}{y^2(x+2y)} \\
& + \frac{2a^2 c^5 e^{-yt} (e^{-xt} - 1)}{xy^2} + \frac{4a^2 c^5 e^{-yt} (e^{-(x+y)t} - 1)}{y^2(x+y)} - \frac{4a^2 c^5 e^{-yt} (e^{-xt} - 1)}{xy^2} \\
& - \frac{4a^4 c^3 e^{-yt} (e^{-(2x+y)t} - 1)}{x(2x+y)(x+y)} + \frac{4a^4 c^3 e^{-yt} (e^{-xt} - 1)}{x^2(x+y)}
\end{aligned}$$

Cont.

$$\begin{aligned}
& + k^3 b^4 \left[\begin{aligned}
& + \frac{4a^4 c^3 e^{-yt} (e^{-(x+y)t} - 1)}{xy(x+y)} - \frac{4a^3 c^4 e^{-yt} (e^{-(x+2y)t} - 1)}{y^2(x+2y)} \\
& + \frac{4a^3 c^4 e^{-yt} (e^{-xt} - 1)}{xy^2} + \frac{8a^3 c^4 e^{-yt} (e^{-(x+y)t} - 1)}{y^2(x+y)} - \frac{8a^3 c^4 e^{-yt} (e^{-xt} - 1)}{xy^2} \\
& - \frac{4a^2 c^5 e^{-yt} (e^{-3yt} - 1)}{3y(x-3y)(x-2y)} + \frac{4a^2 c^5 e^{-yt} (e^{-xt} - 1)}{x(x-3y)(x-2y)} + \frac{4a^2 c^5 e^{-yt} (e^{-xt} - 1)}{xy(x-2y)} \\
& - \frac{4a^3 c^4 e^{-yt} (e^{-(2x+y)t} - 1)}{(2x+y)(2x-y)(x+y)} + \frac{4a^3 c^4 e^{-yt} (e^{-xt} - 1)}{x(2x-y)(x+y)} - \frac{2a^3 c^4 e^{-yt} (e^{-2yt} - 1)}{y(2x-y)(x-2y)} \\
& - \frac{4a^2 c^5 e^{-yt} (e^{-(x+2y)t} - 1)}{xy(x+2y)} + \frac{4a^2 c^5 e^{-yt} (e^{-xt} - 1)}{x^2 y} - \frac{4a^2 c^5 e^{-yt} (e^{-2yt} - 1)}{xy(x-2y)} \\
& + \frac{4ac^6 e^{-yt} (e^{-3yt} - 1)}{3y^2(x-3y)} - \frac{4ac^6 e^{-yt} (e^{-xt} - 1)}{xy(x-3y)} - \frac{2ac^6 e^{-yt} (e^{-2yt} - 1)}{y^2(x-2y)} \\
& + \frac{4ac^6 e^{-yt} (e^{-xt} - 1)}{xy(x-2y)} - \frac{4a^4 c^3 e^{-yt} (e^{-3xt} - 1)}{3x^2(3x-y)} + \frac{4a^4 c^3 e^{-yt} (e^{-yt} - 1)}{xy(3x-y)} \\
& - \frac{4a^4 c^3 e^{-yt} (e^{-yt} - 1)}{xy(2x-y)} + \frac{4a^3 c^4 e^{-yt} (e^{-yt} - 1)}{xy^2} + \frac{4a^3 c^4 e^{-yt} (e^{-2xt} - 1)}{xy(2x-y)} \\
& - \frac{8a^3 c^4 e^{-yt} (e^{-yt} - 1)}{y^2(2x-y)} + \frac{4a^2 c^5 e^{-yt} (e^{-(x+2y)t} - 1)}{(x+2y)(x-2y)(x+y)} - \frac{4a^2 c^5 e^{-yt} (e^{-yt} - 1)}{y(x-2y)(x+y)} \\
& - \frac{2a^2 c^5 e^{-yt} (e^{-2xt} - 1)}{x(2x-y)(x-2y)} + \frac{4a^2 c^5 e^{-yt} (e^{-yt} - 1)}{y(2x-y)(x-2y)} - \frac{4a^3 c^4 e^{-yt} (e^{-3xt} - 1)}{3x(3x-y)(2x-y)} \\
& + \frac{4a^3 c^4 e^{-yt} (e^{-yt} - 1)}{y(3x-y)(2x-y)} + \frac{4a^3 c^4 e^{-yt} (e^{-(x+y)t} - 1)}{x(2x-y)(x+y)} - \frac{4a^3 c^4 e^{-yt} (e^{-yt} - 1)}{xy(2x-y)} \\
& - \frac{4a^2 c^5 e^{-yt} (e^{-(2x+y)t} - 1)}{x^2(2x+y)} + \frac{4a^2 c^5 e^{-yt} (e^{-yt} - 1)}{x^2 y}
\end{aligned} \right]
\end{aligned}$$

Cont.

f_4 Continued

 $+ k^3 b^4$

$$\begin{aligned}
 & - \frac{2a^6 c e^{-yt} (e^{(y-4x)t-1})}{x^2 (4x-y)} + \frac{2a^6 c e^{-yt} (e^{(y-2x)t-1})}{x^2 (2x-y)} \\
 & + \frac{4a^6 c e^{-yt} (e^{(y-3x)t-1})}{x^2 (3x-y)} - \frac{4a^6 c e^{-yt} (e^{(y-2x)t-1})}{x^2 (2x-y)} \\
 & - \frac{8a^5 c^2 e^{-yt} (e^{-3xt-1})}{3xy(x+y)} + \frac{8a^5 c^2 e^{-yt} (e^{(y-2x)t-1})}{y(2x-y)(x+y)} \\
 & + \frac{8a^5 c^2 e^{-yt} (e^{(y-3x)t-1})}{xy(3x-y)} - \frac{12a^5 c^2 e^{-yt} (e^{(y-2x)t-1})}{xy(2x-y)} \\
 & + \frac{2a^4 c^3 e^{-yt} (e^{-(2x+y)t-1})}{y(2x+y)(x-2y)} - \frac{2a^4 c^3 e^{-yt} (e^{(y-2x)t-1})}{y(2x-y)(x-2y)} \\
 & - \frac{4a^4 c^3 e^{-yt} (e^{(y-3x)t-1})}{x(3x-y)(x-2y)} + \frac{4a^4 c^3 e^{-yt} (e^{(y-2x)t-1})}{x(2x-y)(x-2y)} \\
 & - \frac{2a^5 c^2 e^{-yt} (e^{(y-4x)t-1})}{x(4x-y)(2x-y)} + \frac{2a^5 c^2 e^{-yt} (e^{(y-2x)t-1})}{x(2x-y)^2} \\
 & + \frac{2a^5 c^2 e^{-yt} (e^{-2xt-1})}{xy(2x-y)} - \frac{4a^5 c^2 e^{-yt} (e^{(y-2x)t-1})}{y(2x-y)^2} - \frac{8a^4 c^3 e^{-yt} (e^{-3xt-1})}{3x^2(x+y)} \\
 & + \frac{8a^4 c^3 e^{-yt} (e^{(y-2x)t-1})}{x(2x-y)(x+y)} + \frac{8a^4 c^3 e^{-yt} (e^{-2xt-1})}{x^2 y} \\
 & - \frac{8a^4 c^3 e^{-yt} (e^{(y-2x)t-1})}{xy(2x-y)} - \frac{2a^3 c^4 e^{-yt} (e^{-(2x+y)t-1})}{y^2(2x+y)} \\
 & + \frac{2a^3 c^4 e^{-yt} (e^{(y-2x)t-1})}{y^2(2x-y)} + \frac{2a^3 c^4 e^{-yt} (e^{-2xt-1})}{xy^2} \\
 & - \frac{4a^3 c^4 e^{-yt} (e^{(y-2x)t-1})}{y^2(2x-y)} - \frac{4a^5 c^2 e^{-yt} (e^{-3xt-1})}{3x^2(x+y)}
 \end{aligned}$$

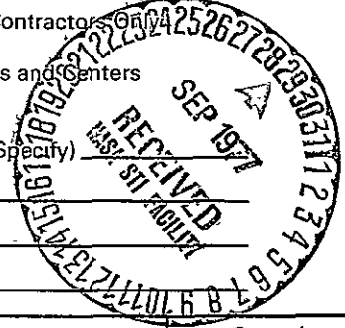
Cont.

f_4 Continued

$$\begin{aligned}
 &+ k^3 b^4 \left[\begin{aligned}
 &+ \frac{8a^2 c^5 e^{-yt} (e^{-(x+y)t} - 1)}{x^2 (x+y)} - \frac{8a^2 c^5 e^{-yt} (e^{-yt} - 1)}{x^2 y} \\
 &- \frac{4ac^6 e^{-yt} (e^{-(x+2y)t} - 1)}{y(x+y)(x+2y)} + \frac{4ac^6 e^{-yt} (e^{-yt} - 1)}{y^2 (x+y)} \\
 &+ \frac{4ac^6 e^{-yt} (e^{-(x+y)t} - 1)}{xy(x+y)} - \frac{12ac^6 e^{-yt} (e^{-yt} - 1)}{xy^2} \\
 &- \frac{2a^3 c^4 e^{-yt} (e^{-(2x+y)t} - 1)}{x^2 (2x+y)} + \frac{2a^3 c^4 e^{-yt} (e^{-yt} - 1)}{x^2 y} \\
 &- \frac{4a^3 c^4 e^{-yt} (e^{-(x+y)t} - 1)}{x^2 (x+y)} - \frac{4a^3 c^4 e^{-yt} (e^{-yt} - 1)}{x^2 y} \\
 &- \frac{8a^2 c^5 e^{-yt} (e^{-(x+2y)t} - 1)}{x(x+y)(x+2y)} + \frac{8a^2 c^5 e^{-yt} (e^{-yt} - 1)}{y^2 (x+y)} - \frac{8a^2 c^5 e^{-yt} (e^{-yt} - 1)}{xy^2} \\
 &+ \frac{2ac^6 e^{-yt} (e^{-3yt} - 1)}{3y^2 (x-2y)} - \frac{2ac^6 e^{-yt} (e^{-yt} - 1)}{y^2 (x-2y)} - \frac{4ac^6 e^{-yt} (e^{-(x+y)t} - 1)}{x(x+y)(x-2y)} \\
 &+ \frac{4ac^6 e^{-yt} (e^{-yt} - 1)}{xy(x-2y)} - \frac{2a^2 c^5 e^{-yt} (e^{-(2x+y)t} - 1)}{x(2x+y)(2x-y)} + \frac{2a^2 c^5 e^{-yt} (e^{-yt} - 1)}{xy(2x-y)} \\
 &+ \frac{2a^2 c^5 e^{-yt} (e^{-2yt} - 1)}{y^2 (2x-y)} - \frac{4a^2 c^5 e^{-yt} (e^{-yt} - 1)}{y^2 (2x-y)} - \frac{8ac^6 e^{-yt} (e^{-(x+2y)t} - 1)}{x(x+y)(x+2y)} \\
 &+ \frac{8ac^6 e^{-yt} (e^{-yt} - 1)}{xy(x+y)} + \frac{4ac^6 e^{-yt} (e^{-2yt} - 1)}{xy^2} - \frac{2c^7 e^{-yt} (e^{-3yt} - 1)}{3y^3} \\
 &+ \frac{2c^7 e^{-yt} (e^{-yt} - 1)}{y^3} + \frac{2c^7 e^{-yt} (e^{-2yt} - 1)}{y^3} - \frac{4c^7 e^{-yt} (e^{-yt} - 1)}{y^3}
 \end{aligned} \right]
 \end{aligned}$$

plus terms having a multiplier with a power of k larger than 3.

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